then $T(\beta)$ spans $R(T)$, so

$$\{T(v_1), \dots, T(v_k), T(v_{k+1}), \dots, T(v_{k+l})\} \quad \text{spans } R(T)$$

Since $T(v_1) = \dots = T(v_k) = 0$, we see that

$$\{T(v_{k+1}), \dots, T(v_{k+l})\} \quad \text{spans } R(T)$$

I claim that this is a basis for $R(T)$; in order to show this, I need to demonstrate that it is LI. Suppose that

$$a_1 T(v_{k+1}) + \dots + a_l T(v_{k+l}) = 0$$

Then $$T(a_1 v_{k+1} + \dots + a_l v_{k+l}) = 0$$

$$\Rightarrow \quad a_1 v_{k+1} + \dots + a_l v_{k+l} \in N(T)$$

Thus $$a_1 v_{k+1} + \dots + a_l v_{k+l} = b_1 v_1 + \dots + b_k v_k$$

since $\{v_1, \dots, v_k\}$ is a basis for $N(T)$.

But $\{v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}\}$ is LI (it's a basis!)

so $a_1 = \dots = a_l = b_1 = \dots = b_k = 0$. In particular, $a_1, \dots, a_l$ are all zero, so $\{T(v_{k+1}), \dots, T(v_{k+l})\}$ is LI.