8

4 : (a)

The auxiliary polynomial is $x^2 - 2ax + a^2+b^2$

which has roots $x = \frac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2}$

$= a \pm ib$

assume $b \neq 0$ throughout

By the Corollary to ~~the~~ Th^{m} 2.33, we know that a basis for the space of solutions in $C^\infty(\mathbb{C})$ is

$$\{ \exp((a+ib)t), \exp((a-ib)t) \}$$

We seek <u>real</u> solutions (ie solutions $y \in C^\infty(\mathbb{C})$ such that $y(t) \in \mathbb{R}$ for all $t \in \mathbb{R}$). We know that this will give us all the solutions in $C^\infty(\mathbb{R})$ because $C^\infty(\mathbb{R})$ is a subspace of $C^\infty(\mathbb{C})$ (where we regard $C^\infty(\mathbb{C})$ as a vector space over $\mathbb{R}$ in the natural way). So we seek

$$(\star) \quad y = (A+iB)\exp((a+ib)t) + ((C+iD)\exp((a-ib)t)$$
$$= e^{at}(A+iB)(\cos(bt) + i\sin(bt)) + e^{a}(C+iD)(\cos(bt) - i\sin(bt))$$

such that $y(t)$ is real for all t. The imaginary part of $(\star)$ is

$$e^{at}\left(A \overset{\sin}{\cancel{\cos}}(bt) + B\cos(bt)\} + D\cos(bt) - C\sin bt\right)$$

For this to be real for all $t \in \mathbb{R}$, we need

$$(A-C)\sin bt + (B+D)\cos(bt) = 0 \quad \text{for all } t$$

$\Rightarrow$ $A = C$, $B = -D$ as the set $\{\cos bt, \sin bt\}$ is LI in $C^\infty(\mathbb{R})$ [if $b \neq 0$]