Then $\quad f = a_1 f_1 + \dots + a_{n-1} f_{n-1} + bh$

$\Rightarrow \quad a_i = f(v_i) = 0 \quad$ as $\quad v_i \in N(f)$

$\Rightarrow \quad f = bh \qquad [b \neq 0$ as $f \neq 0_{V^*}]$

$b' \neq 0$ either

Similarly $\quad g = b'h$, so $\quad f = \alpha g \quad$ where $\quad \alpha = b/b'$.

Case II : $\quad V$ is infinite-dimensional.

If $f = 0_{V^*}$ then we are done (same argument again).

Otherwise $\exists \; x \in V$ with $f(x) \neq 0$ and hence $g(x) \neq 0$.

~~Then~~ Claim : $\quad g(y) = \dfrac{g(x)}{f(x)} f(y) \quad \forall \; y \in V$

Proof : Suppose not. Then $\exists \; y' \in V$ with

$$g(y') \neq \frac{g(x)}{f(x)} f(y')$$

Consider $W = \operatorname{span}\{x, y'\}$ and restrict f and g to give maps $f|_W \in W^*$, $g|_W \in W^*$.

$f|_W(w) = 0 \iff g|_W(w) = 0$, so we can apply Case I to conclude that

$g|_W(w) = \alpha f|_W(w) \quad \forall \; w \in W$

for some $\alpha \in \mathbb{C}$. Putting $w = x$ we see that $\alpha = \dfrac{g(x)}{f(x)}$.

But then $g|_W(y') \neq \alpha f|_W(y')$. $\#$

This proves the Claim.

Taking $\alpha = g(x)/f(x)$ proves Case II. We are done. $\square$