#2 See class notes / textbook.

#3 (a) We will find a basis $\beta = \{p_1(x), p_2(x)\}$ for $P_1(\mathbb{R})$ such that $\{f_1, f_2\}$ is the basis dual to β.

This proves (by Thm 2.24) that $\{f_1, f_2\}$ is a basis for V^*.

Let $p_1(x) = a + bx$
$p_2(x) = c + dx$

We need:

$$f_1(p_1) = 1 \Rightarrow a + b/2 = 1$$
$$f_1(p_2) = 0 \Rightarrow 2a + 2b = 0$$
$$f_2(p_1) = 0 \Rightarrow c + d/2 = 0$$
$$f_2(p_2) = 1 \Rightarrow 2c + 2d = 1$$

So $a = 2,\ b = -2,\ c = -1/2,\ d = 1$

and $\beta = \{2 - 2x,\ -\frac{1}{2} + x\}$.

(b) <u>Case I</u>: V is finite-dimensional.

If $f = 0_{V^*}$ then $g = 0_{V^*}$ too and we can take α = anything.
Otherwise, $R(f) = \mathbb{C}$ so $\text{rank}(f) = 1$
$\Rightarrow \text{nullity}(f) = n - 1$ where $n = \dim V$
(Dimension Theorem)

Pick a basis $\{v_1, \ldots, v_{n-1}\}$ for $N(f)$. Since $N(f) = N(g)$, this is also a basis for $N(g)$. Extend to a basis $\beta = \{v_1, \ldots, v_{n-1}, v\}$ for V.
Let $\beta^* = \{f_1, \ldots, f_{n-1}, h\}$ be the basis for V^* dual to β.