Extending $\{g_1\}$ to an o.n. basis $\gamma = \{\frac{1}{\sqrt{2}}, \sqrt{3/2}\, x\}$ for $P_1(\mathbb{R})$ gives

$$[T]_\beta^\gamma = \begin{pmatrix} \frac{2\sqrt{45}}{\sqrt{19-10\sqrt{2}}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of T is 1, and the non-zero singular value $\sigma_1 = \dfrac{2\sqrt{45}}{\sqrt{19-10\sqrt{2}}}$

[Sorry about the horrible numbers: I made a calculational error when setting the question.]

(d) The only one-dimensional T-invariant subspace of $\mathbb{R}^3$ is span $\{(1,1,1)\}$, so any eigenvector of T must be a scalar multiple of $(1,1,1)$. If T were self-adjoint, there would be an o.n. basis for $\mathbb{R}^3$ consisting of eigenvectors for T, so T is not self-adjoint. However, T is ~~[illegible]~~ length- and angle-preserving, so T is orthogonal.