(c) First let's compute $T^*: P_1(\mathbb{R}) \to P_2(\mathbb{R})$.

This will be easier if we find o.n. bases for $P_2(\mathbb{R})$ and $P_1(\mathbb{R})$ [these won't be the o.n. bases which we are looking for].

Now

$$\langle 1, 1\rangle = 2 \qquad \langle 1, x\rangle = 0 \qquad \langle 1, x^2\rangle = 2/3$$
$$\langle x, 1\rangle = 0 \qquad \langle x, x\rangle = 2/3 \qquad \langle x, x^2\rangle = 0$$
$$\langle x^2, 1\rangle = 2/3 \qquad \langle x^2, x\rangle = 0 \qquad \langle x^2, x^2\rangle = 2/5$$

So applying Gram-Schmidt to the standard basis for $P_2(\mathbb{R})$ gives an o.n. basis $\tilde{\beta} = \{y_1, y_2, y_3\}$ for $P_2(\mathbb{R})$, where:

$$y_1 = \frac{1}{\sqrt{2}}$$

$$y_2 = \frac{1}{\text{length}}\Big(x - \underbrace{\langle x, y_1\rangle}_{\text{zero}} y_1\Big)$$

$$= \sqrt{\tfrac{3}{2}}\, x$$

$$y_3 = \frac{1}{\text{length}}\left(x^2 - \langle x^2, y_2\rangle y_2 - \langle x^2, y_1\rangle y_1\right)$$

$$= \frac{1}{\text{length}}\left(x^2 - \sqrt{2}/3\right)$$

$$= \frac{\sqrt{45}}{\sqrt{38 - 20\sqrt{2}}}\left(x^2 - \sqrt{2}/3\right)$$

Similarly, an o.n. basis $\tilde{\gamma}$ for $P_1(\mathbb{R})$ is $\{z_1, z_2\}$

where $z_1 = 1/\sqrt{2}$ $\qquad z_2 = \sqrt{3/2}\, x$

Now

$$[T]_{\tilde{\beta}}^{\tilde{\gamma}} = \begin{pmatrix} 0 & 0 & \frac{2\sqrt{45}}{\sqrt{19 - 10\sqrt{2}}} \\ 0 & 0 & 0 \end{pmatrix}$$