It remains to show that **if** $\lambda_1 + \dots + \lambda_n = 0$

and $\lambda_1^2 + \dots + \lambda_n^2 = 0$

and $\dots$

and $\lambda_1^n + \dots + \lambda_n^n = 0$

then $\lambda_1^k + \lambda_2^k + \dots + \lambda_n^k = 0$ for $k = n+1, n+2, \dots$

But this is $\mathrm{tr}(T^k)$.

Cayley-Hamilton implies that

$$T^n = \alpha_0 I + \alpha_1 T + \dots + \alpha_{n-1} T^{n-1} \quad \text{for some } \alpha_0, \dots, \alpha_{n-1} \in \mathbb{C}$$

$$\Rightarrow \quad T^{n+1} = \alpha_0 T + \alpha_1 T^2 + \dots + \alpha_{n-1} T^n$$

$$\Rightarrow \quad \mathrm{tr}(T^{n+1}) = \alpha_0 \,\mathrm{tr}(T) + \dots + \alpha_{n-1} \,\mathrm{tr}(T^n)$$

$$= 0$$

Similarly
$$T^{n+2} = \alpha_0 T^2 + \alpha_1 T^3 + \dots + \alpha_{n-1} T^{n+1}$$

$$= \tilde{\alpha}_0 T + \tilde{\alpha}_1 T^2 + \dots + \tilde{\alpha}_{n-1} T^n$$

$$\Rightarrow \quad \mathrm{tr}(T^{n+2}) = 0 \quad \text{too}$$

etc.

So if $\mathrm{tr}(T) = \mathrm{tr}(T^2) = \dots = \mathrm{tr}(T^n) = 0$

then all eigenvalues of T are zero, and hence T is nilpotent.