canonical basis). Then

$$[T]_\beta^\beta = \begin{pmatrix} \lambda_1 & * & * & \cdots & * \\ & \lambda_2 & * & \cdots & * \\ & & \ddots & & * \\ 0 & & & & \lambda_n \end{pmatrix}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues for T (not necessarily distinct). So $tr(T) = \lambda_1 + \cdots + \lambda_n$.

Also $([T]_\beta^\beta)^2 = \begin{pmatrix} \lambda_1^2 & * & \cdots & * \\ 0 & \lambda_2^2 & \ddots & * \\ 0 & 0 & \ddots & * \\ \vdots & \vdots & & \\ 0 & 0 & \cdots 0 & \lambda_n^2 \end{pmatrix}$

so $tr(T^2) = \lambda_1^2 + \cdots + \lambda_n^2$

$\vdots$

$tr(T^n) = \lambda_1^n + \cdots + \lambda_n^n$

Thus if T is nilpotent, so $\lambda_1 = \cdots = \lambda_n = 0$, then $tr(T) = tr(T^2) = \cdots = tr(T^n) = 0$.

Conversely, if $tr(T) = \cdots = tr(T^n) = 0$ then we know that

$$\lambda_1 + \cdots + \lambda_n = 0$$
$$\lambda_1^2 + \cdots + \lambda_n^2 = 0$$
$$\vdots$$
$$\lambda_1^n + \cdots + \lambda_n^n = 0$$