Proof By induction on k

Base case: $k=1$

$$f(t) = |-t| = -t \quad \checkmark$$

Induction step: assume it is true for $k-1$.

Then:

$$f(t) = \begin{vmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & & 0 & -a_1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & -t & \\ 0 & 0 & & 1 & -a_{k-1}-t \end{vmatrix} \quad (k-1)$$

$$= (-t) \begin{vmatrix} -t & 0 & \cdots & 0 & -a_1 \\ 1 & -t & & 0 & -a_2 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & -t & \\ 0 & 0 & & 1 & -a_{k-1}-t \end{vmatrix} \quad (k-2)$$

this looks like exactly the same sort of determinant but now k is smaller by 1 and $a_0 \to a_1$, $a_1 \to a_2$ etc.

$$+ (-1)^k a_0 \begin{vmatrix} 1 & -t & 0 & \cdots & 0 \\ 0 & 1 & -t & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -t \\ 0 & \cdots & \cdots & 0 & 1 \end{vmatrix}$$

expanding along the first row