Thus $\quad Tu_1 + \dots + T^{k-1}u_{k-1} = Tu_1' + \dots + T^{k-1}u_{k-1}' + w'$

and so

$$T^{k-1}u_1 = T^{k-1}u_1' + T^{k-2}(w')$$

$$\Rightarrow \quad T^{k-1}(u_1 - u_1') = T^{k-2}(w')$$

in $\mathcal{U}_2 + T\mathcal{U}_2 + \dots + T^{k-2}\mathcal{U}_2$ $\qquad$ in W'

but $N(T^{k-1}) = (\mathcal{U}_2 + \dots + T^{k-2}\mathcal{U}_2) \oplus W'$ so both sides are zero

$$\Rightarrow \quad u_1 - u_1' \in N(T^{k-1})$$

$$\Rightarrow \quad u_1 = u_1' = 0 \qquad \text{since} \quad V = \mathcal{U} \oplus \mathcal{U}_1 \oplus N(T^{k-1})$$

Repeating, we find that $\quad u_0 = u_1 = \dots = u_{k-1} = 0$

and $\quad u_0' = u_1' = \dots = u_{k-1}' = 0$

$$\Rightarrow \quad w' = 0$$

They are not all supposed to be zero

Thus the sum is direct. $\square$

By induction, we are done.