Recall that $T: V \to V$ satisfies $T^k = 0$ but $T^{k-1} \neq 0$ and $\mathcal{U}$ is a subspace of V such that $\mathcal{U} \cap N(T^{k-1}) = \{0\}$.

We are trying to prove that $V = (\mathcal{U} + T\mathcal{U} + \dots + T^{k-1}\mathcal{U}) \oplus W$ for some T-invariant subspace W.

We set $V = N(T^{k-1}) \oplus \mathcal{U} \oplus \mathcal{U}_1$ for some subspace $\mathcal{U}_1$ and, by induction on k, have shown that

$$V = (\mathcal{U} + T\mathcal{U} + \dots + T^{k-1}\mathcal{U}) + \underbrace{((\mathcal{U}_1 + T\mathcal{U}_1 + \dots + T^{k-1}\mathcal{U}_1) \oplus W')}_{W}$$

for some T-invariant subspace W' of $N(T^{k-1})$.

<u>Claim</u>: This sum is direct

<u>Proof</u>: Otherwise we could find $u_0, \dots, u_{k-1}$ in $\mathcal{U}$ and $u_0', \dots, u_{k-1}'$ in $\mathcal{U}_1$ and $w' \in W'$ not all zero with

$$u_0 + Tu_1 + \dots + T^{k-1}u_{k-1} = u_0' + Tu_1' + \dots + T^{k-1}u_{k-1}' + w'$$

Applying T^{k-1}, and using the facts that $T^k = 0$ and $w' \in N(T^{k-1})$ we find

$$T^{k-1}u_0 = T^{k-1}u_0'$$

$\Rightarrow$ $u_0 - u_0' \in N(T^{k-1})$. But $u_0 - u_0' \in \mathcal{U} \oplus \mathcal{U}_1$ and $V = \mathcal{U} \oplus \mathcal{U}_1 \oplus N(T^{k-1})$ so $u_0 = u_0' = 0$.