1 Section 2.4

1.1 Problem 14

V has basis

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and F^3 has basis $a = (1, 0, 0)$, $b = (0, 1, 0)$, $c = (0, 0, 1)$. A linear transformation is uniquely defined by how it acts on a basis, so define a linear transformation $T : V \to F^3$ by $T(A) = a$, $T(B) = b$, $T(C) = c$ and extending this linearly on linear combinations $\alpha A + \beta B + \gamma C$ for $\alpha, \beta, \gamma \in F$. This is clearly onto since it is onto a basis of F^3 and is linear by definition. If $T(\alpha A + \beta B + \gamma C) = 0$ we have $\alpha a + \beta b + \gamma c = 0 \Rightarrow \alpha = \beta = \gamma = 0$ since $\{a, b, c\}$ is a basis; so $N(T) = 0 \Rightarrow T$ is one-to-one. So T is invertible and therefore an isomorphism.

1.2 Problem 16

$\Phi(aA + cC) = B^{-1}(aA + cC)B = aB^{-1}AB + cB^{-1}CB = a\Phi(A) + c\Phi(C) \Rightarrow \Phi$ is linear. Clearly by the same logic the map $\Psi : M_{n\times n}(F) \to M_{n\times n}(F)$ defined by $\Psi(A) = BAB^{-1}$ is linear, and $\Phi(\Psi(A)) = BB^{-1}ABB^{-1} = A = B^{-1}BAB^{-1}B = \Psi(\Phi(A)) \Rightarrow \Phi\Psi = \Psi\Phi = \mathbf{1} \Rightarrow \Psi = \Phi^{-1}$. Therefore Ψ is invertible.

1.3 Problem 17

1.3.1 (a)

Let $x, y \in T(V_0)$. Then $\exists a, b \in V_0$ s.t. $T(a) = x$ and $T(b) = y$. Since V_0 is a subspace, $z = ca + db \in V_0$ for any $c, d \in F$ so $T(z) = cx + dy \in T(V_0)$. Therefore $T(V_0)$ is a subspace of W.

1.3.2 (b)

Let $\{v_1, \ldots, v_k\}$ be a basis for V_0. Let $w_i = T(v_i)$. Then $\{w_i\}$ spans $T(V_0)$ by definition, and if $\sum c_i w_i = 0$ for $c_i \in F$ then by linearity $T(\sum c_i v_i) = 0 \Rightarrow c_i v_i = 0$ (since T is one-to-one) $\Rightarrow c_i = 0$ for all i. Therefore $\{w_i\}$ is a basis for $T(V_0)$ which shows dim $(V_0) =$ dim $T(V_0)$.

2 Section 2.5

2.1 Problem 6

In each case the matrix Q is the change-of-basis matrix $(Id)_E^\beta$ whose columns are the vectors of β. Here E denotes the standard basis. We find $(L_A)_\beta$ by computing $T(\beta_i)$ for all $1 \leq i \leq \dim V$. $(T = L_A)$

2.1.1 (a)

$T(\beta_1) = (4,2)^t = 6\beta_1 - 2\beta_2$ $T(\beta_2) = (7,3)^t = 11\beta_1 - 4\beta_2$
so

$$(L_A)_\beta = \begin{bmatrix} 6 & 11 \\ -2 & -4 \end{bmatrix}$$

2.1.2 (b)

$T(\beta_1) = (1,3,2)^t = 2\beta_1 - 2\beta_2 + 1\beta_3$ $T(\beta_2) = (0,3,1)^t = 2\beta_1 - 3\beta_2 + 1\beta_3$ $T(\beta_3) = (0,4,2)^t = 2\beta_1 - 4\beta_2 + 2\beta_3$
so

$$(L_A)_\beta = \begin{bmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{bmatrix}$$

2.2 Problem 7

For both parts, choose the basis $\beta = \{(1,m), (-m,1)\}$.

2.2.1 (a)

$T(\beta_1) = \beta_1$ and $T(\beta_2) = -\beta_2$. So

$$(T)_\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

2.2.2 (b)

In this case we take W_1 to be the given line L and W_2 to be the line perpendicular to it. Since $\beta_i \in W_i$ for $i = 1,2$ we have
$T(\beta_1) = \beta_1$ and $T(\beta_2) = 0$. So

$$(T)_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

3 Problem 8

This is evident by the definitions given in the problem: $Q = (I)_{\beta'}^{\beta}$ and $P = (I)_{\gamma'}^{\gamma} implies P^{-1} = (I)_{\gamma}^{\gamma'}$