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Math 121 Solution Set – Problem Set 2

Section 1.6

Problem 31

a

Since $W_1 \cap W_2$ is a subspace contained in W_2 (which is a vector subspace and therefore a vector space), we can choose a basis $\{v_1, \ldots, v_k\}$ for $W_1 \cap W_2$ and extend it to a basis $\{v_1, \ldots, v_n\}$ for W_2. This shows $k \leq n$, i.e. dim $(W_1 \cap W_2) \leq$ dim (W_2).

b

Choose bases $\{v_1, \ldots, v_m\}$ for W_1 and $\{w_1, \ldots, w_n\}$ for W_2. Then the set $\{v_1, \ldots, v_m, w_1, \ldots, w_n\}$ spans $W_1 + W_2$ and by Theorem 1.9 some subset of this is a basis. This implies dim $(W_1 + W_2) \leq m + n$.

Problem 32

a

Let W_2 be the x-axis and W_1 be the plane $z = 0$.

b

Let W_2 be the z-axis and W_1 be the plane $z = 0$.

c

Let W_1 be the plane $x = 0$ and W_2 be the plane $z = 0$. Then each has dimension 2, their intersection is the y-axis, which has dimension 1, and their sum is $\mathbb{R}^3$, which has dimension 3.

Problem 33

a

$\beta_1 \cap \beta_2 = \emptyset$ because $W_1 \cap W_2 = 0$ by definition and 0 is not an element in any basis (any set including 0 is not linearly independent). Let $\beta_1 = \{v_1, \ldots, v_n\}$ and $\beta_2 = \{w_1, \ldots, w_m\}$. Then if $c_1v_1 + \cdots + c_nv_n + d_1w_1 + \cdots + d_mw_m = 0$ we have $c_1v_1 + \cdots + c_nv_n = -d_1w_1 - \cdots - d_mw_m = 0$ since $W_1 \cap W_2 = 0$. Since the $\{v_i\}$ and $\{w_i\}$ are linearly independent sets, this implies $c_i = d_i = 0$, i.e. $\beta_1 \cap \beta_2$ is linearly independent. This is sufficient to show that it is a basis, since from Homework 2, dim $V = m + n$ in this case and so any linearly independent set of $m + n$ elements is a basis.

b

Clearly $V = W_1 + W_2$ so it remains to show $W_1 \cap W_2 = 0$. Let $\beta_1 = \{v_1, \dots, v_n\}$ and $\beta_2 = \{w_1, \dots, w_m\}$ and so any element $x \in W_1 \cap W_2$ can be written as either $c_1v_1 + \cdots + c_nv_n$ or $d_1w_1 + \cdots + d_mw_m$. So we have $c_1v_1 + \cdots + c_nv_n = -d_1w_1 + \cdots + d_mw_m \Rightarrow c_1v_1 + \cdots + c_nv_n - d_1w_1 - \cdots - d_mw_m = 0 \Rightarrow c_i = d_i = 0$ since $\beta_1 \cup \beta_2$ is a linearly independent set. Thus $W_1 \cap W_2 = 0$, as desired.

Problem 34

a

Choose a basis $\{v_1, \dots, v_k\}$ for W_1 and extend it to a basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ for V. Let $W_2 = \text{span } (v_{k+1}, \dots, v_n)$ and then by 33(b), since the bases for W_1 and W_2 are obviously disjoint and their union is a basis for V, $V = W_1 \oplus W_2$.

b

Two possible choices are $W_2 = \{(0, a_2) | a_2 \in \mathbb{R}\}$ and $W_2' = \{(a_2, a_2) | a_2 \in \mathbb{R}\}$

Section 2.1

Problem 9

a

Let $x = (a_1, a_2)$. Then $T(2x) = (1, 2a_2) \neq (2, 2a_2) = 2T(x)$.

b

Let $x = (a_1, a_2)$. Then $T(2x) = (2a_1, 4a_1^2) \neq (2a_1, 2a_1^2) = 2T(x)$.

(c), (d), and (e) are similar.

Problem 13

If $c_1v_1 + \cdots + c_kv_k = 0$ then $T(c_1v_1 + \cdots + c_kv_k) = T(0) = 0$ since T is linear. Also since T is linear, this implies $T(c_1v_1) + \cdots + T(c_kv_k) = 0 \Rightarrow c_1T(v_1) + \cdots + c_kT(v_k) = 0 \Rightarrow c_1w_1 + \cdots c_kw_k = 0 \Rightarrow c_i = 0$ since the w_i are linearly independent. Therefore the v_i are linearly independent.

Problem 19

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $U : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(e_1) = e_1$, $U(e_1) = 2e_1$ and $T(e_2) = U(e_2) = 0$. (Here e_i is the i^{th} standard basis vector, i.e. $e_1 = (1, 0)$, etc.) Then U and T are clearly distinct, and $N(T) = N(U) = \{(x, y) | x = 0\}$ and $R(T) = R(U) = \{(x, y) | y = 0\}$.

Problem 35

a

Since V is finite-dimensional, by the dimension theorem, dim V = dim $R(T)$ + dim $N(T)$. Since $V = R(T) + N(T)$ we know (from HW 2) that dim V = dim $N(T)$ + dim $R(T)$ - dim $R(T) \cap N(T)$ so dim $R(T) \cap N(T) = 0$, i.e. $N(T) \cap R(T) = 0$ and so $V = R(T) \oplus N(T)$.

b

We have dim $N(T) + R(T)$ = dim $R(T)$ + dim $N(T)$ - dim $R(T) \cap N(T)$ = dim $R(T)$ + dim $N(T)$ since dim $R(T) \cap N(T)$ = dim $(0) = 0$. Since V is finite-dimensional, by the dimension theorem dim V = dim $R(T)$ + dim $N(T)$. So dim $R(T) + N(T)$ = dim V, but since $R(T) + N(T) \subset V$ we must have $R(T) + N(T) = V$ and so we have $N(T) \oplus R(T) = V$¿

Section 2.2

Problem 3

Part 1

Recall the definition of the standard basis: $E = \{e_1, \ldots, e_n\}$ where $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, etc. In terms of the standard basis for $\mathbb{R}^3$, $T(e_1) = (1, 1, 2)$ and $T(e_2) = (-1, 0, 1)$. Now we have to write this in terms of the basis $\gamma = \{\gamma_1, \gamma_2, \gamma_3\}$: by inspection, $T(e_1) = -(1/3)\gamma_1 + 0\gamma_2 + (2/3)\gamma_3$ and $T(e_2) = -1\gamma_1 + 1\gamma_2 + 0\gamma_3$ so we have

$$[T]_\beta^\gamma = \begin{bmatrix} -1/3 & -1 \\ 0 & 1 \\ 2/3 & 0 \end{bmatrix}$$

Part 2

If we let $\alpha = \{\alpha_1, \alpha_2\}$, we can also easily check that $T(\alpha_1) = -(7/3)\gamma_1 + 2\gamma_2 + (2/3)\gamma_3$ and $T(\alpha_2) = -(11/3)\gamma_1 + 3\gamma_2 + (4/3)\gamma_3$ and so

$$[T]_\alpha^\gamma = \begin{bmatrix} -7/3 & -11/3 \\ 2 & 3 \\ 2/3 & 4/3 \end{bmatrix}$$

Problem 13

We know from HW 2 that (T, U) is linearly dependent iff one is a constant multiple of each other. So, if T and U are both nonzero, but $R(T) \cap R(U) = 0$, there is some vector x for which $T(x) \neq cU(x)$ for any $c \in \mathbb{F}$, where $\mathbb{F}$ is the underlying field. So (T, U) must be linearly independent.