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Math 121 Solution Set – Problem Set 2

1 Section 1.5

1.1 Problem 9

Let $u, v \in V$, a vector space over a field $\mathbb{F}$. Let $u = cv$, $c \in \mathbb{F}$ i.e. u is a multiple of v. Then $(-1)u+(c)v = 0$, i.e. u and v are linearly dependent. Now let $\{u, v\}$ be linearly dependent. Then there are nonzero constants such that $bu + cv = 0 \Rightarrow u = (-c/b)v$, i.e. u is a multiple of v.

1.2 Problem 13

I will only prove (b), since (a) is similar and easier. Note that proving the statement "$\{u, v, w\}$ is linearly independent if and only if $\{u + v, u + w, v + w\}$ is linearly independent " is equivalent to proving "$\{u, v, w\}$ is linearly dependent if and only if $\{u + v, u + w, v + w\}$ is linearly dependent".

Let $\{u, v, w\}$ be linearly dependent. Then there exist constants, not all nonzero, such that $au + bv + cw = 0$. Then since char $\mathbb{F} \neq 2$ we have

$$\frac{1}{2}(a+b-c)(u+v) + \frac{1}{2}(a-b+c)(u+w) + \frac{1}{2}(-a+b+c)(v+w) = au+bv+cw = 0$$

and the constants on the left side are not all zero since a, b, c are not all zero. So $\{u + v, u + w, v + w\}$ is linearly dependent.

Now let $\{u+v, u+w, v+w\}$ be linearly dependent. Then there are constants, not all nonzero, such that $a(u+v)+b(u+w)+c(v+w) = 0$. This means $(a+b)u+(a+c)v+(b+c)w = 0$ with these constants nonzero since a, b, c are nonzero. So $\{u, v, w\}$ is linearly dependent.

1.3 Problem 18

We will prove this by induction. If S has only one element, it is nonzero by definition and therefore S is a linearly independent set. Now assume it is true for a set of n polynomials. Take a set S of $n + 1$ polynomials and list them in order of increasing degree: $\{f_1, \ldots, f_{n+1}\}$. Then if this set were linearly dependent, there must be coefficients c_i in $\mathbb{F}$ such that $c_{n+1}f_{n+1} = -c_nf_n - \cdots - c_1f_1$ which is impossible if the degree of f_{n+1} is greater than the degree of any of the other f_i. So S is a linearly independent set.

2 Section 1.6

2.1 Problem 2

Since all these sets have three elements, each will necessarily be a basis for $\mathbb{R}^3$ if it is L.I. or if it spans. (a) Write these vectors as $\{u, v, w\}$. By writiing out the equation $au+bv+cw = 0$ as a system of three linear equations, we find $a = -2b$ and $b = (4/5)c$. This implies $(11/5)c = 0 \Rightarrow a = b = c = 0$. Therefore this set is linearly independent. (b) Label these u, v, w. Note that $3u + 4v = w$, so the set is not L.I. and therefore not a basis.

2.2 Problem 14

(a) I claim $\{u = (1, 0, 1, 0, 0), v = (1, 0, 0, 1, 0), w = (0, 1, 0, 0, 0), z = (0, 0, 0, 0, 1)\}$ is a basis for W_1, and so W_1 has dimension 4. To check that it is L.I., write $au + bv + cw + dz = 0$ and it is easy to see (by looking at the last four coordinates) that this implies a = b = c = d = 0. Furthermore, since any element in W_1 can be uniquely described by the last four coordinates (the first is the sum of the third and fourth) it is clear that this set spans W.

(b) I claim $\{u = (0, 1, 1, 1, 0), v = (1, 0, 0, 0, -1)\}$ is a basis for W_2, and so W_2 has dimension 2. As in (a) it is easy to check that these are linearly independent, and since any vector in W_2 is determined by its first two coordinates, we see that $\{u, v\}$ spans and is so a basis.

2.3 Problem 15

I claim that a basis is the matrices given by $\{E_{ij} \cup (E_{ii} - E_{11})\}$ where in the first set we take $i \neq j$ and in the second $2 \leq i \leq n$. (In general E_{ij} has a 1 in the $(i, j)^{\text{th}}$ entry and zeroes elsewhere.) These are clearly linearly independent, since each basis element as a 1 in an entry where every other matrix has zeroes and so any equation of linear dependence implies all coefficients are 0. To see that this spans, note that we can choose the $n^2 - 1$ entries $a_{ij}|(i, j) \neq (1, 1)$ and then a_{11} is uniquely determined. So W has dimension $n^2 - 1$

2.4 Problem 17

A basis for this subspace is the matrices $\{E_{ij} - E_{ji}|n \geq j > i \geq 1$. Checking that it is a basis is similar to 15.

2.5 Problem 21

This is equivalent to proving that a vector space is finite-dimensional iff it does not contain an infinite L.I. subset.

Let V be finite-dimensional. Then it is generated by n elements, and if it contained an infinite L.I. subset it would contain a L.I. subset of m elements with $m > n$. This contradicts the Replacement Theorem (1.10) and so V does not contain an infinite L.I. subset.

Let V not contain an infinite L.I. subset. Then for some m there is no L.I. set of more than m elements. Take a L.I. set B of m elements; since adding any element of V would create a linearly dependent set, B must span V and so it is a basis, i.e. V has dimension m which is finite.

2.6 Problem 29

(a) The first conclusion follows from the second, so as in the hint consider a basis $\{u_1, \ldots, u_k\}$ for $W_1 \cap W_2$ and extend this to a basis $\{u_1, \ldots, u_k, v_1, \ldots, v_m\}$ for W_1 and $\{u_1, \ldots, u_k, w_1, \ldots, w_n\}$ for W_2. It is easy to see that then $\{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_n\}$ is a basis for $W_1 + W_2$. So adding dimensions we have:

$$dim(W_1+W_2) = k+n+m = (k+n)+(k+m)-k = dim(W_1)+dim(W_2)-dim(W_1 \cap W_2)$$

(b) This follows from part (a) due to the observation that the only vector subspace of dimension zero is the zero subspace.

3 Section 1.7

3.1 Problem 3

Using 1.6 Problem 21, we know this is infinite-dimensional if we can construct an infinite L.I. subset. Consider $S = \{1, \pi, \pi^2, \ldots, \pi^n, \ldots\}$. If this is linearly dependent than it has a finite linearly dependent subset, because in an equation of linear dependence only finitely many coefficients can be nonzero. In this case, this gives us a polynomial equations with rational coefficients

$$c_n\pi^n + \cdots + c_1\pi + c_0 = 0$$

which, as in the hint, contradicts the transcendence of π. So this vector space is infinite dimensional.