

Conversely, let $f \in W_1^\circ \cap W_2^\circ \Rightarrow f(a) = f(b) = 0, \forall a \in W_1, b \in W_2$.
 $f(a+b) = f(a) + f(b) = 0 + 0 = 0, \forall a+b \in W_1 + W_2$.
 So $W_1^\circ \cap W_2^\circ \subset (W_1 + W_2)^\circ$.

2.6.15 $T: V \rightarrow W$ linear $\Rightarrow N(T^t) = [R(T)]^\circ$.

$$\begin{aligned} N(T^t) &= \{g \in W^* \mid T^t(g) = T_0\} = \{g \in W^* \mid gT = T_0\} = \\ &= \{g \in W^* \mid (gT)(x) = 0, \forall x \in V\} \\ &= \{g \in W^* \mid g(w) = 0, \forall w = T(x)\} \\ &= \{g \in W^* \mid g(w) = 0, \forall w \in R(T)\} = \\ &= [R(T)]^\circ \text{ by definition.} \end{aligned}$$

2.6.20 (~~Exercise to Problem 2.26~~) (finite-dim. case) $T: V \rightarrow W$ linear.

(a) T onto $\Leftrightarrow T^t$ is one-to-one

\Rightarrow " T onto $\Rightarrow R(T) = W$ "

$$\Rightarrow (R(T))^\circ = \{f \in W^* \mid f(x) = 0, \forall x \in R(T) = W\} = \{0\}$$

\Rightarrow from (15) above, $(R(T))^\circ = N(T^t) = \{0\} \Rightarrow T^t$ is one-to-one. ✓

\Leftarrow " T^t one-to-one $\Rightarrow N(T^t) = \{0\}$ "

\Rightarrow from (5), $[R(T)]^\circ = \{0\}$ also.

$R(T) \subset W$ subspace

$$\Rightarrow \dim(R(T)) + \underbrace{\dim(R(T))^\circ}_4 = \dim W \Rightarrow \dim(R(T)) = \dim W$$

By previous homework problem, this implies $R(T) = W$
 $\Rightarrow T$ is onto. ✓

(b) T^t onto $\Leftrightarrow T$ is one-to-one

First note that $N(T) = \{0\} \Leftrightarrow N((T^t)^t) = \{0\}$.

This is easy to prove if you understood Thm. 2.26.