(c) $(S^\circ)^\circ = \text{Span}(\varphi(S))$

(i) Show $(S^\circ)^\circ \subset \text{Span}(\varphi(S))$.

Let $\hat{x} \in (S^\circ)^\circ$. Suppose $\hat{x} \notin \text{Span}(\varphi(S))$

From (b), $\exists f \in S^\circ$ s.t. $f(x) \neq 0 \Rightarrow \hat{x}(f) = f(x) \neq 0$

but this is a contradiction $\Rightarrow$ $\hat{x} \in \text{Span}(\varphi(S))$. ✓

(ii) Show $\text{Span}(\varphi(S)) \subset (S^\circ)^\circ$.

Let $\hat{x} \in \text{Span}(\varphi(S))$

$\Rightarrow \hat{x} = a_1 \hat{x}_1 + \cdots + a_n \hat{x}_n$ for $x_1, \ldots, x_n \in S$.

$\Rightarrow x \in S$.

Look at $f \in S^\circ$. $\Rightarrow f(x) = 0, \forall x \in S$.

$\hat{x}(f) = f(x) = 0 \Rightarrow \hat{x} \in (S^\circ)^\circ$. ✓

(i), (ii) $\Rightarrow (S^\circ)^\circ = \text{Span}(\varphi(S))$.

(d) $W_1 = W_2 \Leftrightarrow W_1^\circ = W_2^\circ$. (Thanks to Kelly Shue).

The forward implication is absolutely trivial (by defn.).

Now for the converse:

Suppose $W_1 \neq W_2$. Then, without restricting generality, we can

assume $\exists w \in W_1$ s.t. $w \notin W_2$

From part (b), if $w \notin W_2$, $\exists f \in W_2^\circ$ s.t. $f(w) \neq 0$

But $W_1^\circ = W_2^\circ$ So $f \in W_1^\circ$!

This cannot be, since, $\forall f \in W_1^\circ$ should have $f(w) = 0$, $\forall w \in W_1$.

So this is a contradiction and hence $W_1 = W_2$. QED.

(e) $(W_1 + W_2)^\circ = W_1^\circ \cap W_2^\circ$.

Let $f \in (W_1 + W_2)^\circ$. Then $f(a+b) = 0$, $\forall a \in W_1, b \in W_2$.

$= f(a) + f(b)$ (by linearity).

$= 0$. $\Rightarrow f(a) = f(b) = 0$ (Why?)

So $f \in W_1^\circ$ and $f \in W_2^\circ$ $\Rightarrow$ $f \in W_1^\circ \cap W_2^\circ$.

$\Rightarrow (W_1 + W_2)^\circ \subset W_1^\circ \cap W_2^\circ$.