Suppose $a_1T_1 + \cdots + a_nT_n = 0$.

You can now assume that $n \leq \deg(f)$, because we are working in $\mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$. (as opposed to $\mathcal{L}(P_k(\mathbb{R}), P_k(\mathbb{R}))$).

Since $T_1, \ldots, T_n$ now have different degrees, and you've proven in a previous problem set that a set of polynomials of different degree is necessarily linearly independent, $\{T_1, \ldots, T_n\}$ must be linearly independent.

Q.E.D.

$\boxed{2.2.16}$ We can prove an even stronger result:

$T: V \to W$ linear

$\underset{n}{\underset{\shortparallel}{\dim V}} = \underset{n}{\underset{\shortparallel}{\dim W}}$

Suppose $\text{nullity}(T) = k$. Let $\{v_1, \ldots, v_k\}$ be a basis for $N(T)$

Extend this to a basis $\beta = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V.

Let $w_i = T(v_i)$ for all $i = \overline{1,n}$.

Then $\text{Span}\{T(v_1), \ldots, T(v_n)\} = R(T)$

$\dim(R(T)) + \dim(N(T)) = n \Rightarrow \dim(R(T)) = k$ $\Big\} \Rightarrow \{w_1, \ldots, w_k\}$ is a basis for $R(T)$.

We can extend this to a basis $\gamma = \{w_1, \ldots, w_k, w_{k+1}, \ldots, w_n\}$ for W.

This yields the matrix:

$$[T]_\beta^\gamma = \begin{array}{c} \\ (k) \\ \\ n-k \end{array} \begin{array}{c} \begin{array}{cc} (k) & (n-k) \end{array} \\ \begin{bmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & \ddots & & & \\ & & & 1 & & \\ & 0 & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix} \end{array}$$

, which is diagonal.

$\boxed{2.3.9}$ Let U be the transform: $U: F^2 \to F^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 0 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix}$

For instance, T " " $T: F^2 \to F^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ 0 \end{bmatrix}$

$\Rightarrow \begin{cases} UT\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix} \\ TU\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases}$

$\Rightarrow \boxed{UT \neq TU}$.

Let $[U] = A \Rightarrow A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$[T] = B \Rightarrow B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \boxed{AB \neq BA}$.