

2.2.9

$$T: V \rightarrow V$$

$$z \mapsto \bar{z}$$

$V =$ vector space over reals.

i. T is linear: For any $a_1, b_1, a_2, b_2, c \in \mathbb{R}$, we have:

$$T[(a_1 + ib_1) + (a_2 + ib_2)] = T[a_1 + a_2 + i(b_1 + b_2)] =$$

$$= [(a_1 + a_2), -i(b_1 + b_2)] = [a_1 - ib_1, a_2 - ib_2] = T(a_1 + ib_1) + T(a_2 + ib_2).$$

$$T[c(a_1 + ib_1)] = T[ca_1 + icb_1] = ca_1 - icb_1 = c(a_1 - ib_1) = cT(a_1 + ib_1).$$

Note: This only works if c is real. As a counter-example, try it with $c = i, a_1 = 0, b_1 = 1$. So T is not linear if regarded as a v.s. over \mathbb{C} .

ii. Compute $[T]_{\beta}$, $\beta = \{1, i\}$

This will be a 2×2 matrix. See where basis goes under T :

$$\left. \begin{aligned} T(1) = 1 &= 1 \cdot 1 + 0 \cdot i = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ T(i) = -i &= 0 \cdot 1 + (-1) \cdot i = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \end{aligned} \right\} \Rightarrow [T]_{\beta} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

2.2.11

$T: V \rightarrow V$ a linear transf.

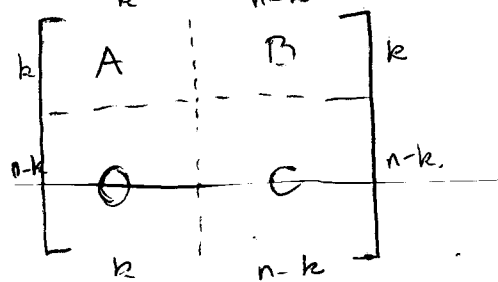
W is T -invariant, $\dim(W) = k$.

Start with a basis $\beta = \{v_1, \dots, v_k\}$ of W , and extend this to a basis $\beta' = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ of V .

Since W is T -invariant, we must have $T(v_i) \in W$ for all $i \in \{1, \dots, k\}$, which means that each $T(v_i)$ can be written as a linear combination of v_1, \dots, v_k alone. That is,

$$T(v_i) = \sum_{j=1}^k a_{ji} v_j + \sum_{j=k+1}^n 0 \cdot v_j,$$

Which gives us the desired form for our matrix.



2.2.14

$$V = \mathcal{P}(\mathbb{R}), T_j(f(x)) = f^{(j)}(x).$$

From example 4 in 2.2, we know that T_j is linear (you can check this inductively or directly). Hence $\{T_1, \dots, T_n\} \subset \mathcal{L}(V)$.

As for linear independence: Each time we take the derivative, the degree of the polynomial drops by 1 (Some people went on to say that this implies that all elements of the set $\{T_1, \dots, T_n\}$ have different degree - this is not

exactly true, as you can have quite a few of degree 0 if $n > \deg(f)$). What is true, however, is that you have the following: