2.2.9 $T: V \to V$ $\quad V =$ vector space over reals.
$z \mapsto \bar{z}$

i. Tis linear: For any $a_1, b_1, a_2, b_2, c \in \mathbb{R}$, we have:

$T[(a_1 + ib_1) + (a_2 + ib_2)] = T[a_1 + a_2 + i(b_1 + b_2)] =$
$= [(a_1 + a_2), -i(b_1 + b_2)] = [a_1 - ib_1, a_2 - ib_2] = T(a_1 + ib_1) + T(a_2 + ib_2)$.

$T[c(a_1 + ib_1)] = T[ca_1 + icb_1] = ca_1 - icb_1 = c(a_1 - ib_1) = cT(a_1 + ib_1)$.

Note: This only works if c is real. As a counter-example, try it with $c = i$, $a_1 = c$, $b_1 = 1$. So T is not linear if regarded as a V.S. over $\mathbb{C}$.

ii. Compute $[T]_\beta$, $\beta = \{1, i\}$

This will be a 2×2 matrix. See where basis goes under T:

$$\left.\begin{array}{l} T(1) = 1 = 1 \cdot 1 + 0 \cdot i = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ T(i) = -i = 0 \cdot 1 + (-1) \cdot i = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \end{array}\right\} \Rightarrow [T]_\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

2.2.11 $T: V \to V$ a linear transf.
W is T-invariant, $\dim(W) = k$.

Start with a basis $\beta = \{v_1, \dots v_k\}$ of W, and extend this to a basis $\gamma = \{v_1, \dots v_k, v_{k+1}, \dots v_n\}$ of V.

Since W is T-invariant, we must have $T(v_i) \subseteq W$ for all $i \in \{1, \dots k\}$, which means that each $T(v_i)$ can be written as a linear combination of $v_1, \dots v_k$ alone. That is, v_i

$$T(v_i) = \sum_{j=1}^{k} a_{ji} v_j + \sum_{j=k+1}^{n} 0 \cdot v_j,$$

which gives us the desired form for our matrix.

$$\begin{array}{c|cc|c} & k & n-k & \\ k & A & B & k \\ n-k & O & C & n-k \\ & k & n-k & \end{array}$$

2.2.14 $V = \mathcal{P}(\mathbb{R})$, $T_j(f(x)) = f^{(j)}(x)$.

From example 4 in 2.2, we know that T_j is linear (you can check this inductively or directly). Hence $\{T_1, \dots T_n\} \subset \mathcal{L}(V)$.

As for linear independence: Each time we take the derivative, the degree of the polynomial drops by 1 (Some people went on to say that this implies that all elements of the set $\{T_1, \dots T_n\}$ have different degree – this is not exactly true, as you can have quite a few of degree 0 if $n > \deg(f)$).

What is true, however, is that you have the following: