HOMEWORK #4 SOLUTIONS

by SIMONA TOPOR POP

$\boxed{2.1.28}$ For each W, we need to show that $T(W) \subseteq W$.

i. $W = \{0\}$. Then $T(0) = 0$ by linearity of $T \Rightarrow T(\{0\}) = \{0\}$.

ii. $W = V$. $T: V \to V$ implies $T(V) \subseteq V$ by defn.

iii. $W = R(T)$. We can look at T as $T: V \to R(T) \subseteq V$. Let $w \in R(T)$. Then $w \in V$ (since $R(T) \subseteq V$) $\Rightarrow T(w) \in R(T) \Rightarrow T(R(T)) \subseteq R(T)$.

iv. $W = N(T)$. Let $w \in N(T) \Rightarrow T(w) = 0$. But $\{0\} \subseteq N(T)$ since $T(0) = 0$ $\Rightarrow$ $T(N(T)) \subseteq N(T)$.

$\boxed{2.1.31}$ $V = R(T) \oplus W$
W T-invariant

a) $W \subseteq N(T)$.

Let $w \in W$ $\Rightarrow T(w) \in W$ since T is invariant.
$\Rightarrow T(w) \in R(T)$ by definition.
But $W \cap R(T) = \{0\}$ by direct sum prop.
$\Rightarrow T(w) = 0 \Rightarrow T(w) \in N(T)$
Hence $W \subseteq N(T)$.

b) V finite dimensional $\Rightarrow W = N(T)$.

Since V is finite-diml., we can apply the dimension formula:

① $\dim(V) = \text{rank}(T) + \text{nullity}(T) = \dim(R(T)) + \dim(N(T))$.

Additionally, we know $V = R(T) + W$ by direct sum formula.

By previous hw. problem,

$\dim(R(T) + W) = \dim(V) = \dim(R(T)) + \dim(W) - \cancel{\dim(R(T) \cap W)}^{0}$

② $\dim(V) = \dim(R(T)) + \dim(W)$.

From ① & ②, we must have: $\dim(N(T)) = \dim(W)$.

Apply Theorem 1.11 to finish off the proof (Friedberg p.50) $\Rightarrow W = N(T)$.

c) Let $V = \mathbb{R}[X]$ be the (infinite) set of polynomials with real coefficients.

and $T: \mathbb{R}[x] \to \mathbb{R}[x]$ given by:

$$f(x) \longmapsto f'(x)$$

Then $R(T) = V$, and $N(T) = \{a \mid a \in \mathbb{R}\}$ (all constant polynomials)

But $W = \{0\}$. Hence $N(T) \neq W$ as desired.