$\boxed{1.2.26}$ Show $M_{m\times n}(F) = W_1 \oplus W_2$.

· From defn, we need: ① • W_1, W_2 subspaces of $M_{m\times n}(F)$ (why is this true?)

② • $W_1 \cap W_2 = \{0\}$. Well, $W_1 = \{A \in M_{m\times n}(F) \mid A_{ij} = 0 \text{ for } i > j\}$.

$W_2 = \{B \in M_{m\times n}(F) \mid B_{ij} = 0 \text{ for } i < j\}$.

The only intersection of these 2 occurs for the zero matrix: $\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}$
(you can show this in detail)

③ • $W_1 + W_2 = V = M_{m\times n}(F)$. ★ There's a double-inclusion to show here.

→ $W_1 + W_2 \subset V$: $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ 0 & 0 & a_{mn} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & 0 & 0 \\ & \ddots & 0 \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$

$\Rightarrow A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12} & \cdots & a_{1n} \\ b_{21} & \ddots & & a_{2n} \\ \vdots & & \ddots & \vdots \\ b_{m1} & \cdots & & a_{mn}+b_{mn} \end{bmatrix} \in M_{m\times n}(F)$ ✓

→ $V \subset W_1 + W_2$: Pick an arbitrary matrix $C \in \cancel{W} = M_{m\times n}(F)$.

$C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mn} \end{bmatrix}$. Write $C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & c_{mn} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ c_{21} & 0 & \ddots & 0 \\ \vdots & & \ddots & \vdots \\ c_{m-1,1} & & & \\ c_{m1} & \cdots & c_{m,n-1} & 0 \end{bmatrix}$

(these are c's)

$\in W_1$ $\qquad$ $\in W_2$ (no one says it can't have zeros on diagonal)

So $C \in W_1 + W_2$ as well.

This completes the proof, and hence, by defn, $W_1 \oplus W_2 = M_{m\times n}(F)$.

$\boxed{1.4.10}$ Pick an arbitrary symmetric matrix $M \in M_{2\times 2}(F)$.

Then M has the form: $\begin{bmatrix} a & c \\ c & b \end{bmatrix}$ with $a, b, c \in F$.

Now show that M can be written as a L.C. of M_1, M_2, M_3.

Consider: $\begin{bmatrix} a & c \\ c & b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} =$

$= a M_1 + a M_2 + a M_3$.

Note that this works because $M_{2\times 2}(F)$ is closed under addition & scalar multiplication. So we conclude that the span of $\{M_1, M_2, M_3\}$ is all of $M_{m\times n}(F)$

Note: some people started with the L.C. $aM_1 + bM_2 + cM_3$. This only shows that the linear combination represents a symmetric matrix, but it doesn't show that $\text{span}\{M_1, M_2, M_3\}$ covers all symmetric matrices. This will be essential for harder proofs!