1.3.12 $W = \{A = M_{m\times n}(F) \mid A_{ij} = 0 \text{ whenever } i > j\}$ is a subspace of $M_{m\times n}(F)$.

Check:

- zero vector: $\begin{bmatrix} 0 & \cdots & 0 \\ 0 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in W$, since it's clearly upper triangular.
- closure under addition: $A + B \in W$.

$\left.\begin{array}{l} A_{ij} = 0 \text{ whenever } i > j \\ B_{ij} = 0 \text{ whenever } i>j \end{array}\right| \Rightarrow [A+B]_{ij} = A_{ij} + B_{ij} = 0$ whenever $i > j$.

- closure under multiplication: $cA \in W$

$A_{ij} = 0$ whenever $i > j \Rightarrow cA_{ij} = [cA]_{ij} = 0$ whenever $i > j$.

1.3.23 W_1, W_2 subspaces of V.

(a) $W_1 + W_2$ is a subspace of V, $W_1, W_2 \subset W_1 + W_2$.

- zero vector: $0 = 0_{W_1} + 0_{W_2}$
- closure under $+$: $w \in W_1 + W_2 \Rightarrow w = w_1 + w_2$ with $w_1 \in W_1$, $w_2 \in W_2$.

$w' \in W_1 + W_2 \Rightarrow w' = w_1' + w_2'$ w/ $w_1' \in W_1$, $w_2' \in W_2$.

$w + w' = w_1 + w_2 + w_1' + w_2' = \underbrace{(w_1 + w_1')}_{\in W_1 \text{ (closed under } +\text{, b/c subspace)}} + \underbrace{(w_2 + w_2')}_{\in W_2 \text{ (closed under } +\text{, b/c subspace)}} \in W_1 + W_2$.

- closure under $\times$: $c \in V$, $cw \in W_1 + W_2$.

$c(w_1 + w_2) = cw_1 + cw_2 \in W_1 + W_2$ follows from closure under $\times$ of W_1, W_2.

$W_1 \subset W_1 + W_2$: $\left.\begin{array}{l}\text{Let } w_1 \in W_1 \\ \text{Know } 0_{W_2} \in W_2\end{array}\right| \Rightarrow$ from previous $w_1 + 0_{W_1} \in (W_1 + W_2) \Rightarrow w_1 \in (W_1 + W_2)$

$W_2 \subset W_1 + W_2$: $\left.\begin{array}{l}\text{Let } w_2 \in W_2 \\ \text{Know } 0_{W_1} \in W_1\end{array}\right| \Rightarrow$ from previous, $0_{W_1} + w_2 \in (W_1 + W_2) \Rightarrow w_2 \in (W_2 + W_2)$.

(b) $W \subset V$ contains both W_1 & $W_2 \Rightarrow W_1 + W_2 \subset W$.

Let $u \in W_1 + W_2 \Rightarrow u$ can be written as: $u = w_1 + w_2$ with $w_1 \in W_1$, $w_2 \in W_2$.

$W_1 + W_2$ is a subspace, and so is W - so it's closed under addition.

$\left.\begin{array}{l} W_1 \subset W \Rightarrow w_1 \in W \\ W_2 \subset W \Rightarrow w_2 \in W \end{array}\right| \Rightarrow w_1 + w_2 \in W$. Hence we've concluded $W_1 + W_2 \subset W$.

Note: It's extremely important to start with $u \in W_1 + W_2$ an arbitrary vector, and then step into $u = w_1 + w_2$ for the implication in this problem.