§ 2.6 #10

(a) Since V^* is $(n+1)$-dimensional, to show that $\{f_0, \ldots, f_n\}$ is a basis for V^* it suffices to show that it is LI.

Suppose that

$$a_0 f_0 + \cdots + a_n f_n = 0 \quad \text{----(1)} \quad \text{where} \quad a_0, \ldots, a_n \in \mathbb{F}$$

Applying (1) to the polynomial $(x-c_1)(x-c_2)\ldots(x-c_n)$ we find that

$$a_0(c_0-c_1)(c_0-c_2)\ldots(c_0-c_n) + a_1 \cdot 0 + \cdots + a_n \cdot 0 = 0$$

Since the scalars $c_0, \ldots, c_n$ are distinct, $(c_0-c_1)\ldots(c_0-c_n) \neq 0$ and so a_0 must be zero.

Similarly, applying (1) to $(x-c_0)(x-c_2)(x-c_3)\ldots(x-c_n)$ we conclude that $a_1 = 0$, and so on.

Thus $a_0 = a_1 = \cdots = a_n = 0$, and so $\{f_0, \ldots, f_n\}$ is LI.

Thus $\{f_0, \ldots, f_n\}$ is a basis for V^*.

(b) The Corollary to Theorem 2.26 implies that there is a basis $\{p_0, \ldots, p_n\}$ for V for which $\{f_0, \ldots, f_n\}$ is the dual basis. In other words, $f_i(p_j) = \delta_{ij}$

this notation means "1 if $i=j$ and 0 if $i \neq j$"

i.e. $p_j(c_i) = \delta_{ij}$

$\Rightarrow p_i(c_j) = \delta_{ij}$

(c) We want to find a polynomial $q(x) \in V$ such that $q(c_0) = a_0, \ldots, q(c_n) = a_n$. But $\{p_0, \ldots, p_n\}$ is a basis for V, so if q exists we can write it