MATH 118 : SPRING 1999
SOLUTIONS TO PROBLEM SET 5

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All exercises are from [Dev92].

Exercise 7.17. *Prove that if $x > 1$ or $x < 0$, then $F_4^n(x) \to -\infty$ as $n \to \infty$.*

Proof. If $x > 1$, then $F_4(x) < 0$, so we may assume without loss of generality that $x < 0$. We use the Mean Value Theorem:

$$F_4(x) = F_4(x) - F_4(0) = F_4'(z) \cdot x$$

for some $z \in (x, 0)$. Now $F'(z) = 4 - 8z$, which, if $z < 0$, is bounded below by 4. Hence $F_4(x) < 4x$, meaning $F_4^n(x) < 4^n x$, and the sequence on the right clearly diverges to $-\infty$. □

Exercise 7.18. *Prove that if $x \in [0, 1]$, then $F_4^n(x) \not\to -\infty$ as $n \to \infty$.*

Proof. We note that $F_4'(x) = 4 - 8x$ is nonnegative for $x \in [0, \frac{1}{2}]$ and nonpositive for $x \in [\frac{1}{2}, 0]$. Hence F_4 is monotone on each of these intervals. Thus $F_4[0, \frac{1}{2}] = F_4[\frac{1}{2}, 1] = [0, 1]$. □

Exercise 7.20. *Consider the function $S(x) = \pi \sin x$. Prove that S has at least 2^n periodic points of period n in the interval $[0, 1]$.*

We retreat a bit for some general theory. Borrowing from [CE80, Ch. II.1], let's extract the vital information about S:

Definition. A continuous map $f: [a, b] \to [a, b]$ is *unimodal* if

(a) f has a unique global maximum point c with $f(c) = b$.
(b) f is strictly increasing on $[a, c]$ and strictly decreasing on $[c, b]$.

Furthermore, let's call f *unimodal with fixed boundary* if $f(a) = f(b) = a$.

Food for thought: What kinds of conjugacies preserve unimodality? Certainly monotone ones do. This means that we can always assume via an affine transformation that $[a, b] = [0, 1]$. But we can do even better. We can use a piecewise-linear transformation to make sure that $c = \frac{1}{2}$.

We already know lots of unimodal maps. For instance, F_4, the "tent map" T, and S. These in fact are all unimodal with fixed boundary. F_4 and S are also C^1*-unimodal*, but we don't need that.

We will prove a general fact, from which Exercise 7.20 follows.

Proposition. *Let $f: [a, b] \to [a, b]$ be unimodal with fixed boundary. Then f has at least 2^n points of period n in the interval $[a, b]$.*

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Proof. We will assume $a = 0$, $b = 1$, $c = \frac{1}{2}$ for simplicity. We claim that for each n there exists a partition of the interval [0,1]

$$0 = \alpha_0 < \beta_1 < \alpha_1 < \cdots < \beta_{2^{n-1}} < \alpha_{2^{n-1}} = 1 \tag{1}$$

such that

(i) $f^n(\alpha_i) = 0$ for all i;
(ii) $f^n(\beta_i) = 1$ for all i.

Then it follows that f^n has a fixed point in each interval $[\alpha_{i-1}, \beta_i]$ and one in each $[\beta_i, \alpha_i]$. This gives at leat 2^n n-periodic points for f in $[0, 1]$, as desired.

The proof of the claim is by induction. For the base case $n = 1$, take $\alpha_0 = 0$, $\beta_1 = \frac{1}{2}$, $\alpha_1 = 1$. Now suppose for a given n a partition like (1) is given. Focus on a particular subinterval $[\alpha_{i-1}, \beta_i]$. Since $f(\alpha_{i-1}) = 0$ and $f(\beta_i) = 1$, there must exist γ_i between them such that $f^n(\gamma_i) = \frac{1}{2}$. We can insert δ_i between β_i and α_i in the same way. Then

$$f^{n+1}(\alpha_i) = f^{n+1}(\beta_i) = 0$$

for all i, and

$$f^{n+1}(\gamma_i) = f^{n+1}(\delta_i) = f(\tfrac{1}{2}) = 1$$

for all i, too. So by making our new alphas equal to the union of the old alphas and betas, and making our new betas equal to the union of the gammas and deltas, ordering and renumbering, we have a desired partition for $n + 1$. □

Remark. Careful attention to where f^n is increasing and decreasing, based on where f is increasing and decreasing, can help to replace the words "at least" by "exactly."

Exercise 11.2. *Can a continuous function on $\mathbb{R}$ have a periodic point of period 176 but not one of period 96? Why?*

Proof. $176 = 2^4 \cdot 11$ and $96 = 2^5 \cdot 3$, so $176 \triangleright 96$ in the Šarkovskii ordering. So a continuous function with a period-176 point must have a period-96 point. □

Exercise 11.4. *The graphs in Figure 1 each have a cycle of period four given by* $\{0, 1, 2, 3\}$. *One of these functions has cycles of all other periods, and one has only periods one, two, and four. Identify which function has each of these properties.*

Proof. Let F be the first of these functions. We can recover the function from the graph:

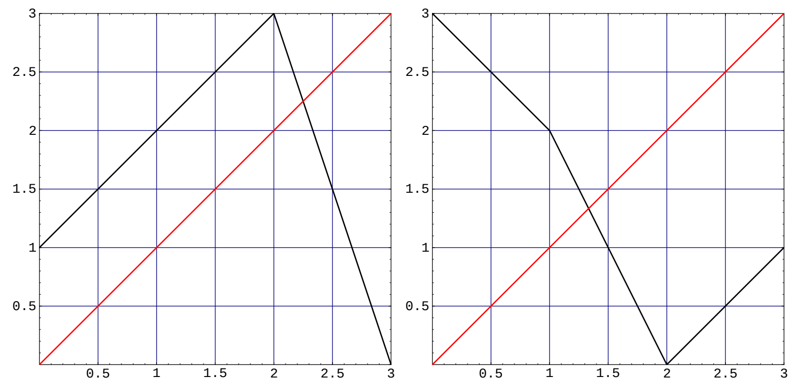
$$F(x) = \begin{cases} x + 1 & 0 \le x \le 2; \\ -3x + 9 & 2 \le x \le 3. \end{cases}$$

Then $\frac{11}{4} \mapsto \frac{3}{4} \mapsto \frac{7}{4} \mapsto \frac{11}{4}$ is a three-cycle. So F has orbits of all periods. See Figure 2(a) for the three-cycle.

Alternatively, we could note that the given four-cycle is of the kind given in Exercise 11.5. So by that problem we have periodic points of all periods.

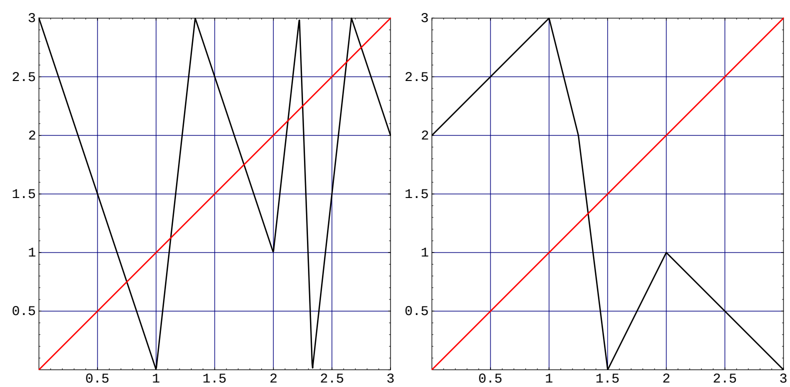
Now we turn to the function G given in Figure 1(b).

$$G(x) = \begin{cases} -x + 3 & 0 \le x \le 1; \\ -2x + 4 & 1 \le x \le 2; \\ x - 2 & 2 \le x \le 3. \end{cases}$$



(a) F (b) G

FIGURE 1. Two graphs with four-cycles. Refer to Exercise 11.4.



(a) F^3 (b) G^3

FIGURE 2. The search for three-cycles in Exercise 11.4.

If G has orbits of any periods besides four, two, and one, it will have an orbit of period eight by Šarkovskii's theorem. So we will prove that G has no eight-cycle.

Notice that $G[0,1] = [2,3]$ and $G[2,3] = [0,1]$. Moreover, $G^4|_{[0,1]} = \mathrm{Id}$, so no eight-cycles can appear on these intervals. But on $[1,2]$, G is completely linear, so it can have at most one fixed point and no periodic points of higher order. See Figure 3. □

Exercise 11.5. *Suppose a continuous function F has a cycle of period $n \geq 3$ given by $a_1 < a_2 < \cdots < a_n$. Suppose that F permutes the a_i according to the rule $a_1 \mapsto a_2 \mapsto \cdots \mapsto a_n$. What can you say about other cycles for F?*

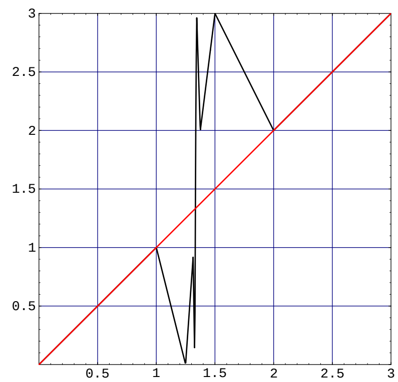


FIGURE 3. The graph of G^8. Notice the only fixed points of G^8 are periodic of order less than 8.

Proof. We claim that F has a three-cycle, and thus has period points of all other periods.

Asume $n > 3$, because the $n = 3$ case is already directly taken care of. Let $I_0 = [a_{n-1}, a_n]$. Then

$$F(I_0) = [a_1, a_n] \supset I_0,$$

so there exists $I_1 \subset I_0$ such that $F(I_1) = I_0$. Now

$$F^2(I_1) = F(I_0) = [a_1, a_n] \supset [a_1, a_{n-1}],$$

so there exists $I_2 \subset I_1$ such that $F^2(I_2) = [a_1, a_{n-1}]$. Finally,

$$F^3(I_2) = F[a_1, a_{n-1}] \supset [a_{n-1}, a_n] = I_0 \supset I_1 \supset I_2,$$

so there exists $p \in I_2$ such that $F^3(p) = p$.

We claim p is of prime period three. If it isn't, it must be a fixed point (the only other divisor of three is one), and therefore must also satisfy $F^2(p) = p$. But $p \in I_2 \subset [a_{n-1}, a_n]$ and $F^2(p) \in F^2(I_2) = [a_1, a_{n-1}]$. The only possibility is that $p = a_{n-1}$, but this point is is period $n > 3$, a contradiction. So p must be of prime period three. □

Exercise 11.6. *Consider the piecewise linear graph in Figure 4. Prove that this function has a cycle of period seven but not of period five.*

Proof. The cycle $1 \mapsto 4 \mapsto 5 \mapsto 3 \mapsto 6 \mapsto 2 \mapsto 7 \mapsto 1$ is there by design. To prove that H has no four-cycle, we track subintervals as in Devaney.

Note first that

$$[1, 2] \to [4, 7] \to [1, 5] \to [3, 7] \to [1, 6] \to [2, 7]$$

under H, so $H^5[1, 2] \cap [1, 2] = \{2\}$. This means if a five-cycle lives in $[1, 2]$, it must be 2. But 2 is a seven-cycle.

The same process keeps working:

$$[2, 3] \to [6, 7] \to [1, 2] \to [4, 7] \to [1, 5] \to [3, 7]$$

So if a five-cycle exists in this subinterval, it must be 3, which isn't a five-cycle.

$$[3, 4] \to [5, 6] \to [2, 3] \to [6, 7] \to [1, 2] \to [4, 7]$$

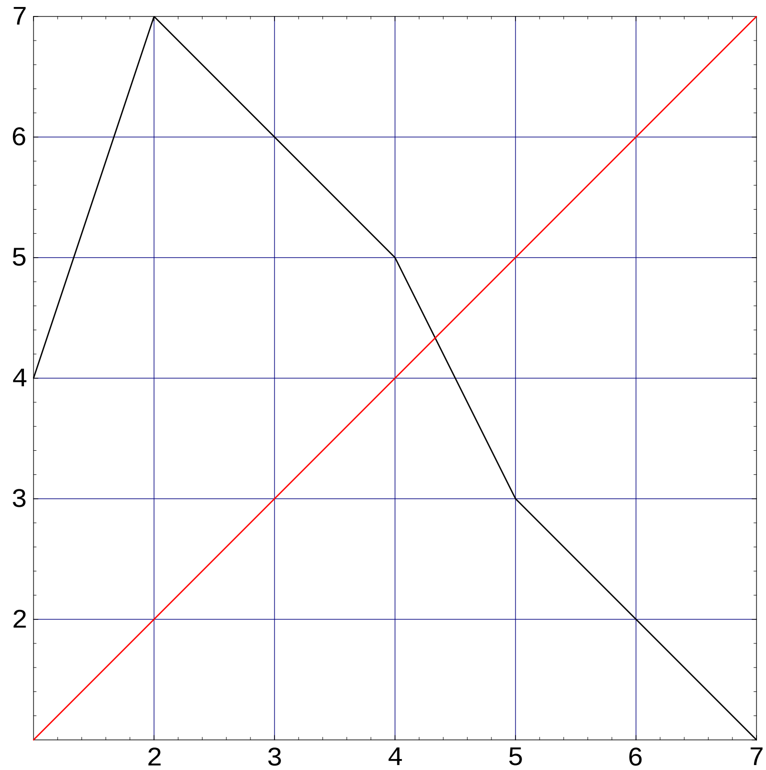


FIGURE 4. The function H for Exercise 11.6.

So if a five-cycle exists in this subinterval, it must be 4, which isn't a five-cycle. Things get a little tricky at $[4,5]$:

$$[4,5] \to [3,5] \to [3,6] \to [2,6] \to [2,7] \to [1,7],$$

so we could have a five-cycle here (there is a nontrivial intersection at the fixed point). But this is the five-fold composition of decreasing maps, which must be decreasing, so any periodic point must be unique and of period one, so we are out of luck here, too. Finally,

$$[5,6] \to [2,3] \to [6,7] \to [1,2] \to [4,7] \to [1,5]$$

$$[6,7] \to [1,2] \to [4,7] \to [1,5] \to [3,7] \to [1,6]$$

so no five-cycles can live in these subintervals either. We have exhausted all possibilities; no five-cycle can exist. See Figure 5 for the proof-by-picture. □

REFERENCES

[CE80] Pierre Collet and Jean-Pierre Eckmann, *Iterated Maps on the Interval as Dynamical Systems*, Birkhaüser, 1980.

[Dev92] Robert L. Devaney, *A First Course in Chaotic Dynamical Systems: Theory and Experiment*, Addison-Wesley, 1992.

[Rob95] Clark Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, CRC Press, 1995.

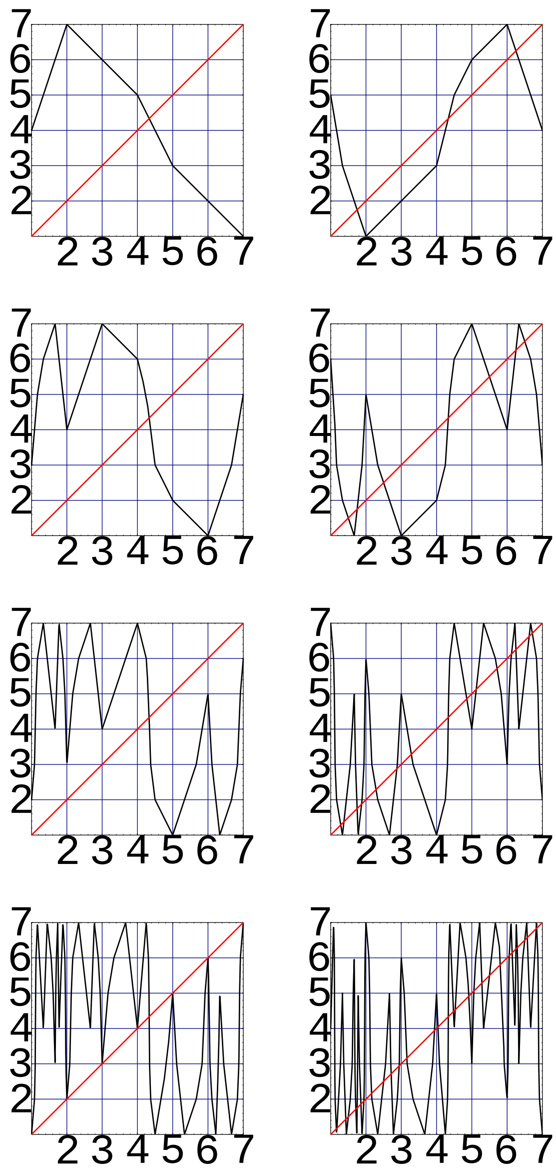


FIGURE 5. The graphs of H^n for $n = 1, 2, \ldots, 8$, where H is as in Exercise 11.6. Notice that there are cycles of prime period one, two, four, and eight, but not of three or five. This agrees (thankfully) with Šarkovskii's Theorem.

[SG99] Shlomo Z. Sternberg and Daniel Goroff, *Math 118 Notes*, Harvard University Department of Mathematics, February 1999, Preliminary Version.