

**MATH 118 : SPRING 1999**  
**SOLUTIONS TO PROBLEM SET 4**

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A WORD ABOUT RIGOR

Many of you have noticed that [Dev89] is not all that rigorous of a textbook. It does have lots and lots of examples and exercises and pictures, things that [Ste99] lacks. But the level at which we want this course to be appreciates both. Hence, even though problems come from the easier text, they should be worked out according to the standards of the harder one.

Many students are using a term found in Devaney called “graphical analysis.” This is a fancy term for “proof by picture.” Such proofs rely on the assumption that the given picture is a completely accurate and generic depiction of the situation at hand. This cannot be categorically justified. Thus, while graphical analysis is a good way to understand why a proposition is true, it does not constitute a proof.

Even more egregious is the proof by *Mathematica*. As in, “I graphed this function on *Mathematica* and observed that all values were less than 2; therefore the function is bounded by 2.” This is probably true but you may not deduce it. *Mathematica* graphs a function by choosing a finite set of points and interpolating them. This can lead to false conclusions, especially if the function has many critical points (or has discontinuities, etc.). Again, graphing is a useful method for figuring out how to tackle the problem, but it does not suffice.

For these problems on bifurcations, I expected the following: Either an explicit showing of the period-doubling or fold (and whether it’s attracting), or a computation with derivatives to prove that such must occur, as in Propositions 2.2.1 and 2.2.2 of [Ste99] or from similar propositions proven in class. In some cases (viz., Exercise 1(h) below), the former is difficult or impossible. Pictures and graphics are great visual evidence for the veracity of your claims, but I did not give full credit for picture proofs.

Let me also mention that the term “neutral fixed point” is not being used in this course. The definitions of attracting and repelling fixed point are topological rather than differential. Coining a term “neutral fixed point” begs us to set up a trichotomy among attracting, repelling or neutral, but then we immediately violated that by proving that neutral fixed points can sometimes be attracting or repelling! So, in your work, if  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a differentiable map with  $f(p) = p$  and  $|f'(p)| = 1$ , then call  $p$  a *nonhyperbolic* fixed point. One could call a fixed point “ambiguous” if it attracts on one side and repels on the other. But this much different from “indifferent” or “neutral”!

With that, we return you to your regularly scheduled solution set.

## THE CHAIN RULE

Let  $F(\mu, x)$  be either a map from a subset of  $\mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$  or a one-parameter family of maps  $F_\mu: \mathbb{R} \rightarrow \mathbb{R}$ . The notions are interchangeable. Suppose that  $F(0, 0) = 0$  and  $|\frac{\partial F}{\partial x}(0, 0)| = 1$ . Then we have some possibilities for behavior at this fixed point.

First, we could have a fold or saddle-node bifurcation. [Ste99] shows that sufficient conditions for such a bifurcation are:

- (a)  $\frac{\partial F}{\partial x}(0, 0) = 1$ ;
- (b)  $\frac{\partial^2 F}{\partial x^2}(0, 0) > 0$ ;
- (c)  $\frac{\partial F}{\partial \mu}(0, 0) > 0$ .

Notice from the proof (or from the first sentence in the notes after the proof) that the signs of the derivatives in (b) and (c) are irrelevant. We only care that they are nonzero. But this is the most we can relax; as you saw in Exercise 1(h), condition (a) can be satisfied *without* a fold bifurcation happening.

A second possibility is the period-doubling bifurcation. Sufficient conditions for this are given in [Ste99] as

- (d)  $\frac{\partial F}{\partial x}(0, 0) = -1$ ;
- (e)  $\frac{d\lambda}{d\mu}(0) > 0$ ; and
- (f)  $\frac{\partial^3 F}{\partial x^3} > 0$ .

Here  $\lambda$  is the function  $\lambda(x) = \frac{\partial F}{\partial x}(\mu, x(\mu))$ , where  $x(\mu)$  is the solution for  $x$  in terms of  $\mu$  satisfying  $F(\mu, x(\mu)) \equiv 0$  in a neighborhood of zero. Translating (e) into a condition on  $F$  gives

$$\frac{1}{2} \frac{\partial^2 F}{\partial x^2}(0, 0) \frac{\partial F}{\partial \mu}(0, 0) + \frac{\partial^2 F}{\partial x \partial \mu}(0, 0) > 0.$$

Proving this is a good application of *Mathematica's* symbolic power. Again, we note that the  $>$  sign isn't necessary, only a  $\neq$ .

The final condition needs to be enhanced. The point of assuming condition (f) is to conclude that  $\frac{\partial^3 H}{\partial x^3}(0, 0) < 0$ , where  $H(\mu, x) = F(\mu, F(\mu, x)) - x$ . Again translating this back to  $F$ , we get that

$$\frac{\partial^3 H}{\partial x^3}(0, 0) = -3 \left( \frac{\partial F}{\partial x}(0, 0) \right)^2 - 2 \frac{\partial^3 F}{\partial x^3}(0, 0).$$

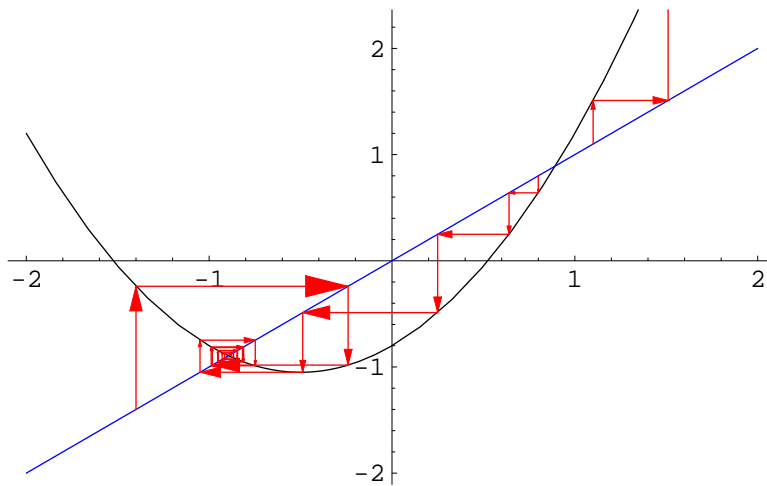
Now you see why (f) is sufficient. But it what suffices is actually less; we only need the quantity above to be nonzero. This helps us especially in the cases of quadratic maps, which have zero third derivatives but can still undergo bifurcations.

Finally, now, the solutions.

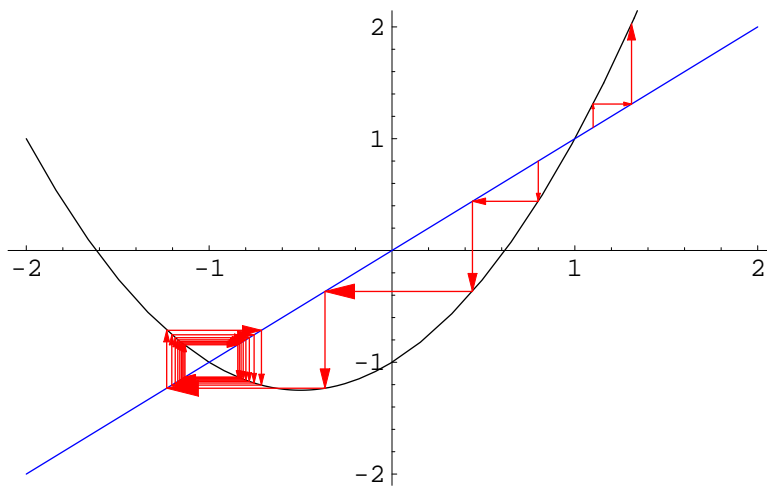
**Exercise 6.1.** *Each of the following functions undergoes a bifurcation of fixed points at the given parameter value. In each case, identify this bifurcation as either a saddle-node or period-doubling bifurcation, or neither of these.*

- (b)  $F_\lambda(x) = x + x^2 + \lambda$  near  $\lambda = -1$ .

We can solve this one explicitly:  $F_\lambda(x)$  has two fixed points  $\pm\sqrt{-\lambda}$  when  $\lambda$  is negative, and the one at  $(\lambda, x) = (-1, -1)$  is nonhyperbolic. The two points of period two which are “born” at  $\lambda = -1$  are  $-1 \pm \sqrt{-1 - \lambda}$ , and since  $(F_\lambda^2)'$  at either of these is  $5 + 4\lambda$ , they are attracting as long as  $-\frac{3}{2} < \lambda < 1$ .



(a)



(b)

FIGURE 1. Graphs and “bouncing diagrams” of  $F_\lambda(x) = x + x^2 + \lambda$  for (a)  $\lambda = -0.8$  and (b)  $\lambda = -1$ . Note the slowness of convergence in (b).

The hypotheses of Proposition 2.2.2 are also satisfied, with the relaxed condition (f). In either case, we can conclude a period-doubling bifurcation. See Figures 1 and 2.

(c)  $G_\mu(x) = \mu x + x^3$  near  $\mu = -1$ .

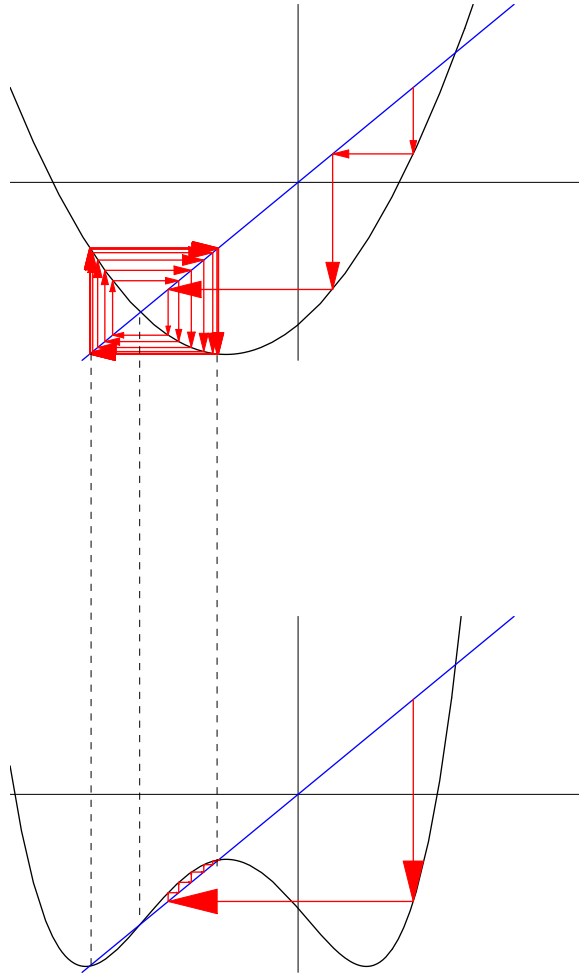


FIGURE 2. Bouncing diagrams for  $F_{-1,2}$  in Exercise 1(b) and its double iterate. Note how the two-cycle in  $F_{-1,2}$  corresponds to a one-cycle (fixed point) in  $F_{-1,2}^2$ .

The fixed points of  $G_\mu$  are 0 and  $\pm\sqrt{1-\mu}$ . As  $\mu$  approaches -1 from above, the fixed point zero becomes repelling and we give birth to the two-cycle  $\pm\sqrt{-1-\mu}$ . Since the multiplier along this cycle is  $(3+2\mu)^2$ , we have attraction as long as  $-2 < \mu < -1$ .

Again, using calculus, we can compute in a matter of three lines that conditions (d), (e) and (f) are satisfied. We have period-doubling.

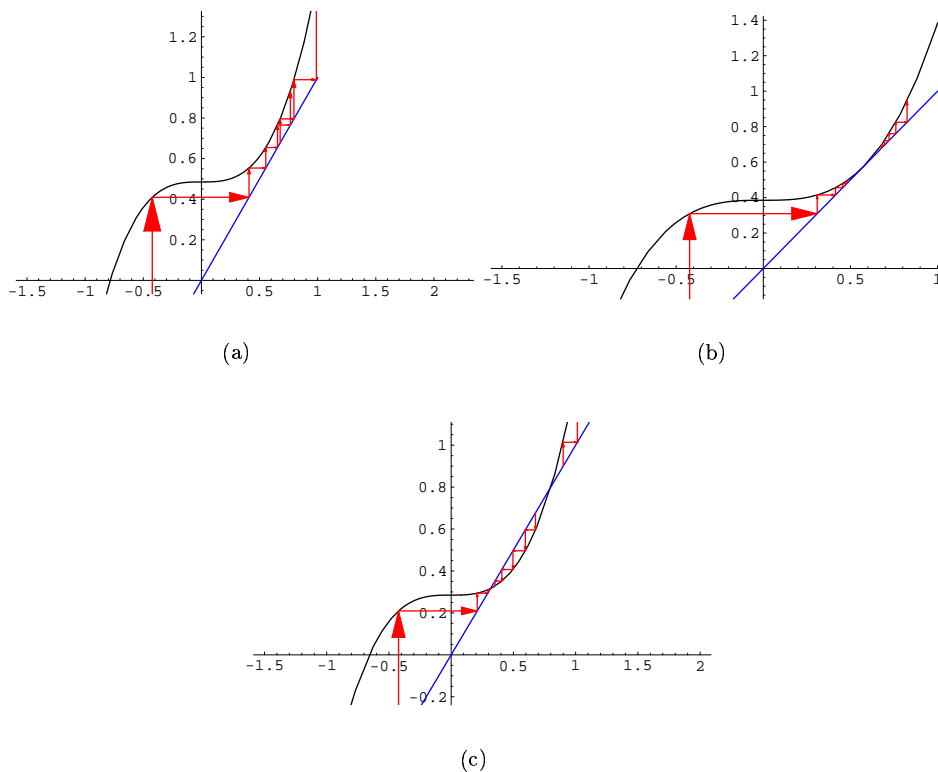


FIGURE 3. Bouncing diagrams for the functions  $G_\mu(x) = \mu x + x^3$  near  $\mu = -1$ . (a)  $\mu = -0.8$  (b)  $\mu = -1$  (c)  $\mu = -1.2$ . Notice the saddle-node bifurcation.

(g)  $F_c(x) = x^3 + c$  near  $c = \frac{2}{3\sqrt{3}}$ .

Solving  $F_c(x) = x$  involves solving the a cubic equation in  $x$ , which, though always possible, is not always trivial. We rely on our differential methods. Let  $c_0 = \frac{2}{2\sqrt{3}}$ . Then  $x_0 = \pm \frac{1}{\sqrt{3}}$  is a fixed point with multiplier 1. Since

$$\frac{\partial^2 F}{\partial x^2}(c_0, x_0) = 6x_0 > 0$$

and

$$\frac{\partial F}{\partial c}(c_0, x_0) = 1 > 0,$$

we have a saddle-node bifurcation by Proposition 2.2.1. See Figure 3.

(h)  $E_\lambda(x) = \lambda(e^x - 1)$  near  $\lambda = -1$ .

This time the algebra is impossible. Luckily we see that zero is a fixed point for each  $E_\lambda$ . Also note that that  $E'_\lambda(0) = \lambda$ , so as  $\lambda$  nears  $-1$ , this becomes a candidate for a period-doubling bifurcation. Since

$$\frac{1}{2} \frac{\partial^2 E}{\partial x^2}(-1, 0) \frac{\partial E}{\partial \lambda}(-1, 0) + \frac{\partial^2 E}{\partial x \partial \lambda}(-1, 0) = (-1)(0) + (-1) = -1$$

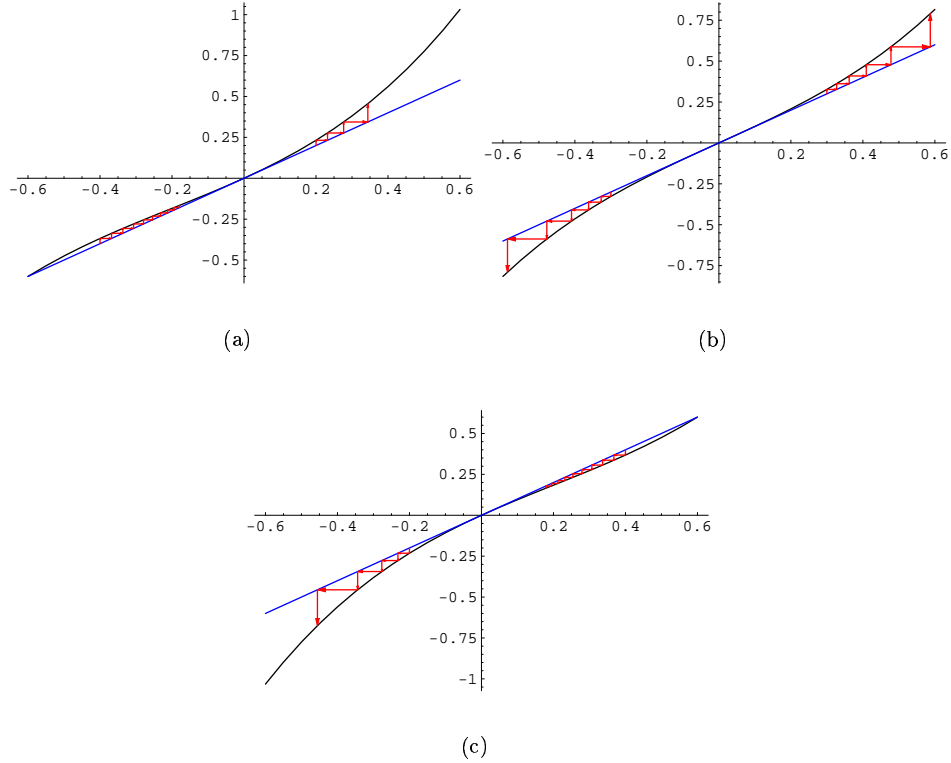


FIGURE 4. Bouncing diagrams for the functions  $F_c(x) = c^2 + x^3$  near  $c = 0$ . (a)  $\mu = -0.8$ . We have a repelling negative fixed point and an ambiguous one (b)  $\mu = -1$ . There is only fixed point, which is slowly repelling. (c)  $\mu = -1.2$ . There is the same fixed point zero which is ambiguous again, and the negative repelling fixed point has moved to the positive side.

and

$$\frac{\partial E^3}{\partial x^3}(-1, 0) = -1$$

we indeed do have a period-doubling bifurcation.

(k)  $F_c(x) = x + c^2 + x^3$  near  $c = 0$ .

Zero is a fixed point of each  $F_c$  and  $F'_c(0) = 1$ . So we might have a saddle-node bifurcation. However,

$$\frac{\partial^2 F}{\partial x^2}(0, 0) = 0$$

so we can't conclude that. We see instead that there is another fixed point at  $x = -c$ , which has multiplier  $1 + c^2$ . As  $c$  nears 0, this fixed point passes through the one at zero, and is repelling only for  $x = 0$ , in which case it is only nonhyperbolic. No wonder things are so problematic. See Figure 4

**Exercise 6.16.** Consider the family of functions  $F_\lambda(x) = x^5 - \lambda x^3$ . Discuss the bifurcation of 2-cycles that occurs when  $\lambda = 2$ .

*Discussion.* Solving  $F_\lambda(x) = x$  gives us 0 and the two real roots

$$p_\pm = \pm \sqrt{\frac{1}{2} \left( \lambda + \sqrt{\lambda^2 + 4} \right)}.$$

0 is superattracting and  $p_\pm$  are both repelling.

But we were asked to discuss two-cycles. We show the graph of  $F_\lambda^2$  for several values of  $\lambda$  (Figure 5). The graph suggests and calculus verifies that this is a fold bifurcation of the two-cycle.

Turning up  $\lambda$  too quickly results in another pair of fixed points emerging. We see this already at  $\lambda = 2.125$ .  $\square$

**Exercise 3.** Discuss the behavior of  $G_\mu(x) = \mu x - x^3$  at  $\mu = 0$  and  $\mu = 2$ .

*Proof.* First let  $\mu$  be near 0. Solving for the fixed points of  $G_\mu$  gives 0 and  $\pm\sqrt{-1 + \mu}$ , which is not real until  $\mu = 1$ . Moreover,  $F'_\mu(0) = \mu$ , so we can conclude: zero is an attracting fixed point of  $F_\mu$  if  $|\mu| < 1$ , and is superattracting for  $\mu = \pm 1$ .

As we increase  $\mu$  to 2, we have to stop at 1. This is where the fixed point 0 becomes repelling, and calculus verifies that we have a bifurcation here. Two new fixed points are born. Finally, at  $\mu = 2$ , we get another bifurcation, this time a period-doubling one. See Figure 6.  $\square$

*Remark.* All of this is in a giant *Mathematica* notebook on the web site. Take a look if you want good examples of code.

#### REFERENCES

- [Dev89] Robert L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley, 1989.
- [Ste99] Shlomo Z. Sternberg, *Math 118 notes*, Harvard University Department of Mathematics, February 1999, Preliminary Version.

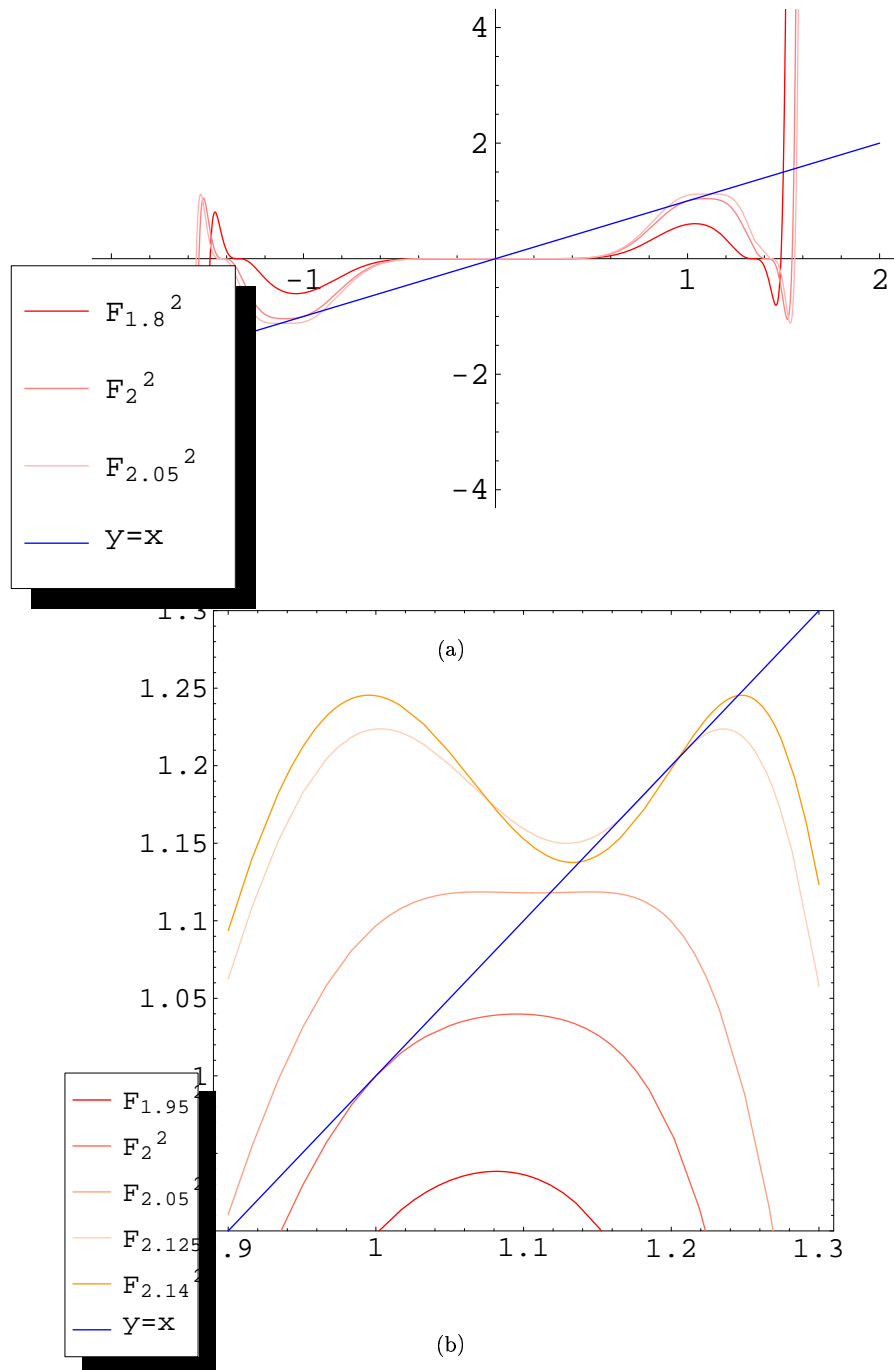
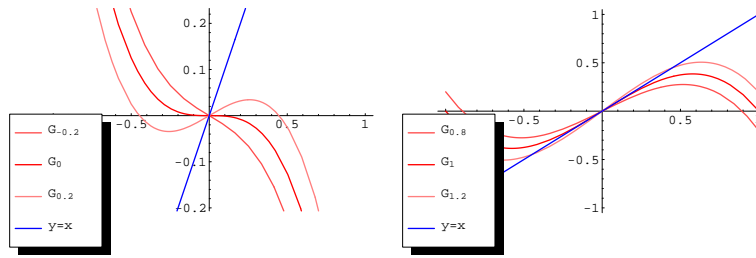
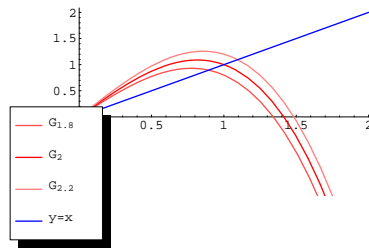


FIGURE 5. The graph of  $F_\lambda^2$  for several values of  $\lambda$  near 2. Notice that we get a saddle-node bifurcation of  $F_\lambda^2$  at 2, then another pair of two-cycles arise.



(a)

(b)



(c)

FIGURE 6. Graphs of  $G_{\mu}(x) = \mu x - x^3$  near some critical values. (a)  $\mu = 0$ , a superattracting fixed point. (b)  $\mu = 1$ , a double saddle-node bifurcation. (c)  $\mu = 2$ , a period-doubling bifurcation.