Problem Set 4:Bifurcations

Math 118−−Dyamical Systems

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■ Theory

We are concerned with families $F(\mu,x)$ of maps of the real line. We assume that $F(0,0)=0$ is a fixed point for F_0:

```
In[1]:= zero = {μ -> 0, x -> 0}
        fixedpoint = {F[0, 0] -> 0}

Out[1]= {μ → 0, x → 0}

Out[2]= {F[0, 0] → 0}
```

For a bifurcation to happen, there is first the condition that the fixed point is *nonhyperbolic*, i.e, that the absolute value of the derivative at the fixed point is plus or minus one.

```
In[3]:= Φa[F_][μ_, x_] :=
         Derivative[0, 1][F][μ, x]
```

■ Fold Bifurcations

```
In[4]:= tangent = {Derivative[0, 1][F][0, 0] -> 1}

Out[4]= {F^(0,1)[0, 0] → 1}

In[5]:= Φb[F_][μ_, x_] :=
         Derivative[0, 2][F][μ, x]

        General::spell1 :
         Possible spelling error: new symbol name "Φb" is similar to existing symbol "Φa".

In[6]:= Φc[F_][μ_, x_] :=
         Derivative[1, 0][F][μ, x]

        General::spell :
         Possible spelling error: new symbol name "Φc" is similar to existing symbols {Φa, Φb}.

In[7]:= P[μ_, x_] := F[μ, x] - x

In[8]:= P[0, 0]

Out[8]= F[0, 0]

In[9]:= % /. fixedpoint

Out[9]= 0
```

```
In[10]:= Derivative[1, 0][P][0, 0]
```

Out[10]= $F^{(1,0)}[0, 0]$

That this is nonzero is precisely condition (c). So we can solve $P(\mu(x), x) = 0$ for μ a function of x in a neighborhood of the origin. Furthermore,

```
In[11]:= Solve[Dt[P[μ, x], x] == 0, Dt[μ, x]]
```

Out[11]= $\left\{\left\{Dt[\mu, x] \to -\frac{-1 + F^{(0,1)}[\mu, x]}{F^{(1,0)}[\mu, x]}\right\}\right\}$

```
In[12]:= nondeg = %[[1]]
```

Out[12]= $\left\{Dt[\mu, x] \to -\frac{-1 + F^{(0,1)}[\mu, x]}{F^{(1,0)}[\mu, x]}\right\}$

In[13]:= $\frac{-1 + F^{(0,1)}[\mu, x]}{F^{(1,0)}[\mu, x]}$ /. zero

Out[13]= $\frac{-1 + F^{(0,1)}[0, 0]}{F^{(1,0)}[0, 0]}$

```
In[14]:= % //. fixedpoint
         % //. tangent
```

Out[14]= $\frac{-1 + F^{(0,1)}[0, 0]}{F^{(1,0)}[0, 0]}$

Out[15]= 0

In[16]:= $D\left[\frac{-1 + F^{(0,1)}[\mu, x]}{F^{(1,0)}[\mu, x]}, x\right]$

Out[16]= $\frac{F^{(0,2)}[\mu, x]}{F^{(1,0)}[\mu, x]} - \frac{(-1 + F^{(0,1)}[\mu, x])\, F^{(1,1)}[\mu, x]}{F^{(1,0)}[\mu, x]^2}$

```
In[17]:= % /. zero
         % //. fixedpoint
         % //. tangent
```

Out[17]= $\frac{F^{(0,2)}[0, 0]}{F^{(1,0)}[0, 0]} - \frac{(-1 + F^{(0,1)}[0, 0])\, F^{(1,1)}[0, 0]}{F^{(1,0)}[0, 0]^2}$

Out[18]= $\frac{F^{(0,2)}[0, 0]}{F^{(1,0)}[0, 0]} - \frac{(-1 + F^{(0,1)}[0, 0])\, F^{(1,1)}[0, 0]}{F^{(1,0)}[0, 0]^2}$

Out[19]= $\frac{F^{(0,2)}[0, 0]}{F^{(1,0)}[0, 0]}$

And this is nonzero by conditions (b) and (c). So in a neighborhood of 0, $\mu(x)$ looks like a parabola, giving two fixed points on one side of μ=0 and none on the other.

Now consider the function $\frac{\partial F}{\partial x}(\mu(x), x)$. We can differentiate this function with respect to x and get

```
In[20]:= Dt[D[F[μ, x], x], x]
```

Out[20]= $F^{(0,2)}[\mu, x] + Dt[\mu, x]\, F^{(1,1)}[\mu, x]$

```
In[21]:= % /. nondeg
```

Out[21]= $F^{(0,2)}[\mu, x] - \frac{(-1 + F^{(0,1)}[\mu, x])\, F^{(1,1)}[\mu, x]}{F^{(1,0)}[\mu, x]}$

```
In[22]:= % /. zero
```

Out[22]= $F^{(0,2)}[0, 0] - \frac{(-1 + F^{(0,1)}[0, 0])\, F^{(1,1)}[0, 0]}{F^{(1,0)}[0, 0]}$

```
In[23]:= % /. tangent
```

Out[23]= $F^{(0,2)}[0, 0]$

By assumption (b), this thing is nonzero, so $\frac{\partial F}{\partial x}(\mu(x), x)$ is monotone in a neighborhood of the origin while $\frac{\partial F}{\partial x}(0, 0) = 1$. This says that one fixed point has multiplier <1 and one has multiplier > 1. This completes the proof of Proposition 2.2.1.

Let's test it: We know experimentally (see Section 6.1 of Devaney) that the family

```
In[24]:= Q[c_, x_] := x^2 + c
```

has a saddle−node bifurcation at c=1/4, x=1/2.

```
In[25]:= {Φa[Q][c, x],
          Φb[Q][c, x],
          Φc[Q][c, x]}
```

Out[25]= $\{2x, 2, 1\}$

```
In[26]:= % /. {c -> 1 / 4, x -> 1 / 2}
```

Out[26]= $\{1, 2, 1\}$

■ Period−Doubling Bifurcations

```
In[27]:= orthogonal = {Derivative[0, 1][F][0, 0] -> -1}
```

Out[27]= $\{F^{(0,1)}[0, 0] \to -1\}$

```
In[28]:= λ[μ_] := D[F[μ, x], x]

In[29]:= Dt[λ[μ], μ]
```

Out[29]= $Dt[x, \mu]\, F^{(0,2)}[\mu, x] + F^{(1,1)}[\mu, x]$

We want to find $\lambda'(0)$.

```
In[30]:= Solve[
          Dt[F[μ, x] - x, μ] == 0, Dt[x, μ]]
```

Out[30]= $\left\{\left\{Dt[x, \mu] \to -\frac{F^{(1,0)}[\mu, x]}{-1 + F^{(0,1)}[\mu, x]}\right\}\right\}$

```
In[31]:= %% /. %[[1]]
```

Out[31]= $-\frac{F^{(0,2)}[\mu, x]\, F^{(1,0)}[\mu, x]}{-1 + F^{(0,1)}[\mu, x]} + F^{(1,1)}[\mu, x]$

```
In[32]:= % /. zero
         % /. orthogonal
```

Out[32]= $-\frac{F^{(0,2)}[0, 0]\, F^{(1,0)}[0, 0]}{-1 + F^{(0,1)}[0, 0]} + F^{(1,1)}[0, 0]$

Out[33]= $\frac{1}{2} F^{(0,2)}[0, 0]\, F^{(1,0)}[0, 0] + F^{(1,1)}[0, 0]$

So this is Sternberg's condition (e): that thing above must be positive. Let's code it into an operator.

```
In[34]:= Φe[F_][μ_, x_] :=
          (1 / 2) Derivative[0, 2][F][μ, x] *
           Derivative[0, 1][F][μ, x] +
           Derivative[1, 1][F][μ, x]

        General::spell :
         Possible spelling error: new symbol name "Φe" is similar to existing symbols {Φa, Φb, Φc}.
```

Finally, we have a condition on the third derivative of *F*. Looking at the proof, we create a function

```
In[35]:= H[μ_, x_] := F[μ, F[μ, x]] - x
```

Notice that

```
In[36]:= H[0, 0]
         % //. fixedpoint

Out[36]= F[0, F[0, 0]]

Out[37]= 0
```

```
In[38]:= D[H[μ, x], x]
         % /. zero
         % //. fixedpoint
         % //. orthogonal
```

Out[38]= $-1 + F^{(0,1)}[\mu, x]\, F^{(0,1)}[\mu, F[\mu, x]]$

Out[39]= $-1 + F^{(0,1)}[0, 0]\, F^{(0,1)}[0, F[0, 0]]$

Out[40]= $-1 + F^{(0,1)}[0, 0]^2$

Out[41]= 0

```
In[42]:= D[H[μ, x], {x, 2}]
         % /. zero
         % //. fixedpoint
         % //. orthogonal

Out[42]= F^(0,1)[μ, F[μ, x]] F^(0,2)[μ, x] + F^(0,1)[μ, x]^2 F^(0,2)[μ, F[μ, x]]

Out[43]= F^(0,1)[0, F[0, 0]] F^(0,2)[0, 0] + F^(0,1)[0, 0]^2 F^(0,2)[0, F[0, 0]]

Out[44]= F^(0,1)[0, 0] F^(0,2)[0, 0] + F^(0,1)[0, 0]^2 F^(0,2)[0, 0]

Out[45]= 0
```

So $H(\mu, x)$ and its first two derivatives vanish at the origin. However,

```
In[46]:= D[H[μ, x], {x, 3}]
         % /. zero
         % //. fixedpoint
         % //. orthogonal

Out[46]= 3 F^(0,1)[μ, x] F^(0,2)[μ, x] F^(0,2)[μ, F[μ, x]] + F^(0,1)[μ, F[μ, x]] F^(0,3)[μ, x] +
          F^(0,1)[μ, x]^3 F^(0,3)[μ, F[μ, x]]

Out[47]= 3 F^(0,1)[0, 0] F^(0,2)[0, 0] F^(0,2)[0, F[0, 0]] + F^(0,1)[0, F[0, 0]] F^(0,3)[0, 0] +
          F^(0,1)[0, 0]^3 F^(0,3)[0, F[0, 0]]

Out[48]= 3 F^(0,1)[0, 0] F^(0,2)[0, 0]^2 + F^(0,1)[0, 0] F^(0,3)[0, 0] + F^(0,1)[0, 0]^3 F^(0,3)[0, 0]

Out[49]= -3 F^(0,2)[0, 0]^2 - 2 F^(0,3)[0, 0]
```

If we assume that this quantity above is nonzero, then we can apply the implicit function theorem to

$$P(\mu, x) = \frac{H(\mu, x)}{x - \mu(x)}$$

Sterberg's condition (f) is that $F^{(0,3)}[0, 0]$ be negative, and that is certainly sufficent but not neccessary. Another sufficient condition is that the entire quantity above is nonzero. Let's code that up into a functional:

```
In[50]:= Φf[F_][μ_, x_] :=
          -3 (Derivative[0, 2][F][μ, x])^2
           - 2 Derivative[0, 3][F][μ, x]

        General::spell :
         Possible spelling error: new symbol name "Φf" is similar to existing symbols {Φa, Φb, Φc, Φe}.
```

Finally, we test the theory on the quadratic family at c = −3/4.

```
In[51]:= Solve[Q[-3 / 4, x] == x, x]

Out[51]= {{x → -1/2}, {x → 3/2}}

In[52]:= D[Q[-3 / 4, x], x] /. %

Out[52]= {-1, 3}
```

```
In[53]:= {Φa[Q][-3/4, -1/2],
          Φe[Q][-3/4, -1/2],
          Φf[Q][-3/4, -1/2]}

Out[53]= {-1, -1, -12}
```

Notice here that $\frac{\partial^3 Q}{\partial x^3}(\frac{-3}{4}, \frac{-1}{4}) = 0$, so the third−derivative criterion does not help. We need the full derivative criteron $\Phi_f(Q)(\frac{-3}{4}, \frac{-1}{4}) \neq 0$. So our modified Proposition 2.2.2 does in fact predict a period−doubling bifurcation.

■ Experiment

Let's apply all of this theory to the problems assigned from Devaney. Even better, we can use the graphical powers of *Mathematica* to visualize our experiments.

■ Nasty code

All of this creates the graphics primitives for the pretty pictures. Except for the packages Arrow, Colors, and Spline, I wrote it. I'm working on putting all of this into a full package. But it's not done yet...

```
In[54]:= << Graphics`Arrow`

In[55]:= << Graphics`Colors`

In[56]:= << Graphics`Spline`

In[57]:= << Graphics`Legend`

In[58]:= << "~/math/118/Bounce.m"

In[59]:= ? Bounce*

        BouncePath[f,x,n,opts] creates a path (polygonal line with arrowheads) that alternates
          between the diagonal y=x and the curve y=f(x).  Starting at {x,x}, the path goes
          to {x,f(x)},{f(x),f(x)},{f(x),f^2(x)}, .... Options are passed to Arrow.

In[60]:= ? BouncePath

        BouncePath[f,x,n,opts] creates a path (polygonal line with arrowheads) that alternates
          between the diagonal y=x and the curve y=f(x).  Starting at {x,x}, the path goes
          to {x,f(x)},{f(x),f(x)},{f(x),f^2(x)}, .... Options are passed to Arrow.

In[61]:= stutter[l_List, n_Integer?NonNegative] :=
          Flatten[Transpose[{Drop[l, -n], Drop[l, n]}], 1]

In[62]:= interp[{x1_, x2_}, type_:Bezier, opts___] :=
          Spline[
           {{x1, 0},
            {(x1 + x2) / 2, (x2 - x1) / (2 GoldenRatio)},
            {x2, 0}},
           type,
           opts]

In[63]:= BP1D[f_, x_, n_Integer?NonNegative] :=
          Map[interp[#, Cubic]&,
           Partition[stutter[NestList[f, x, n], 1], 2]]
```

```
In[64]:= BouncePathEC[anything___] :=
          Drop[BouncePath[anything], -1]

In[65]:= BounceDiagram[
          f_,
          {x_Symbol, xmin_?NumericQ, xmax_?NumericQ},
          pn : {{_?NumericQ, _Integer?NonNegative}..},
         opts___] :=
          With[
           {gtemp = Plot[Evaluate[f[x]], {x, xmin, xmax},
               DisplayFunction -> Identity]},
           Show[gtemp,
            Graphics[{Blue,
              Line[{{xmin, xmin}, {xmax, xmax}}],
              Red,
              Map[BouncePathEC[Function[x, f[x]], #[[1]], #[[2]],
                HeadScaling -> Relative]&,
               pn]}],
            DisplayFunction -> $DisplayFunction,
            opts]]

In[66]:= PhaseDiagram[
          f_,
          {x_Symbol, xmin_?NumericQ, xmax_?NumericQ},
          pn : {{_?NumericQ, _Integer?NonNegative}..},
         opts___] :=
          Show[Graphics[{
             Red,
             Map[
              BP1D[Function[x, f[x]],
               N[#[[1]]], #[[2]]]&,
              pn]}],
           Axes -> {True, False}, opts]
```

■ Exercise 1b: $F_\lambda(x) = x + x^2 + \lambda$

■ Theory

```
In[67]:= F[λ_, x_] := x + x^2 + λ

In[68]:= F_λ_[x_] := F[λ, x]
```

We focus on the particular parameter value λ=−1.

```
In[69]:= Solve[F_λ[x] == x, x]

Out[69]= {{x → -I √λ}, {x → I √λ}}

In[70]:= % /. {λ -> -1}

Out[70]= {{x → 1}, {x → -1}}
```

Taking derivatives,

```
In[71]:= D[F_-1[x], x] /. %

Out[71]= {3, -1}
```

So that negative fixed point is a candidate for a period−doubling bifurcation.

```
In[72]:= {Φa[F][-1, -1],
          Φe[F][-1, -1],
          Φf[F][-1, -1]}
```

Out[72]= $\{-1, -1, -12\}$

So according to Proposition 2.2.2, we do get a period−doubling. Let's verify this.

■ Algebraic verification

```
In[73]:= Solve[Fλ[Fλ[x]] == x, x]
```

Out[73]= $\left\{\left\{x \to -1 - \sqrt{-1-\lambda}\right\}, \left\{x \to -1 + \sqrt{-1-\lambda}\right\}, \left\{x \to -I\sqrt{\lambda}\right\}, \left\{x \to I\sqrt{\lambda}\right\}\right\}$

If $\lambda < -1$, the points of prime period two are real, and

```
In[74]:= D[Fλ[Fλ[x]], x] /. Take[%, 2]
```

Out[74]= $\left\{1 + 2\left(-1 - \sqrt{-1-\lambda}\right) + 2\left(1 + 2\left(-1 - \sqrt{-1-\lambda}\right)\right)\left(-1 + \left(-1 - \sqrt{-1-\lambda}\right)^2 - \sqrt{-1-\lambda} + \lambda\right),\right.$
$\left.1 + 2\left(-1 + \sqrt{-1-\lambda}\right) + 2\left(1 + 2\left(-1 + \sqrt{-1-\lambda}\right)\right)\left(-1 + \left(-1 + \sqrt{-1-\lambda}\right)^2 + \sqrt{-1-\lambda} + \lambda\right)\right\}$

```
In[75]:= Simplify[%]
```

Out[75]= $\{5 + 4\lambda, 5 + 4\lambda\}$

```
In[76]:= Solve[%[[1]] == -1, λ]
```

Out[76]= $\left\{\left\{\lambda \to -\frac{3}{2}\right\}\right\}$

they are stable as long as $-3/2<\lambda<-1$.

■ Graphical Verification

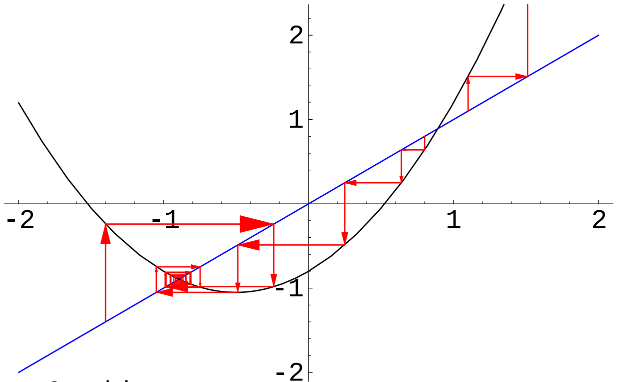
This is where *Mathematica* had a seg fault and stole two hours of my life, as well as all my pretty pictures. :−(

But we shall overcome...

```
In[77]:= BounceDiagram[
           F-0.8,
           {x, -2, 2},
           {{0.8, 10},
            {-1.4, 10},
            {1.1, 2}}]
```

Out[77]= -Graphics-

```
In[77]:= BounceDiagram[
           F-0.8,
           {x, -2, 2},
           {{0.8, 10},
            {-1.4, 10},
            {1.1, 2}}]
```

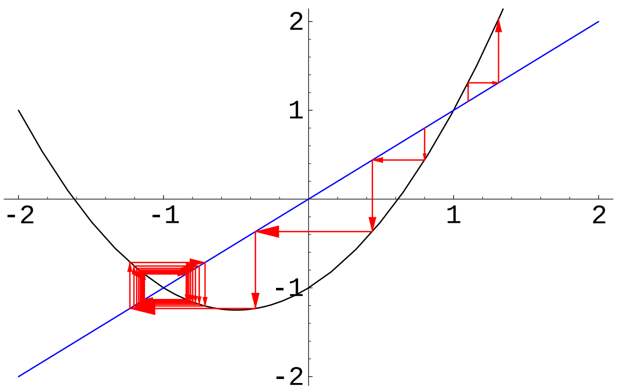



```
Out[79]= - Graphics -
Out[77]= - Graphics -

In[80]:= PhaseDiagram[
In[78]:= PhaseDiagram[
           F-0.8,
           {x, -2, 2},
           {{0.8, 10},
            {-1.4, 10},
            {1.1, 2}},
         PlotRange -> All]

Out[80]= - Graphics -
Out[78]= - Graphics -
```

So here our negative fixed point is attracting, but critically so. Now let’s decrease further to $\lambda=-1.2$.
So at this value of λ the negative fixed point is attracting. There also happens to be a positive one which is repelling. Let’s decrease λ to the critical value -1.

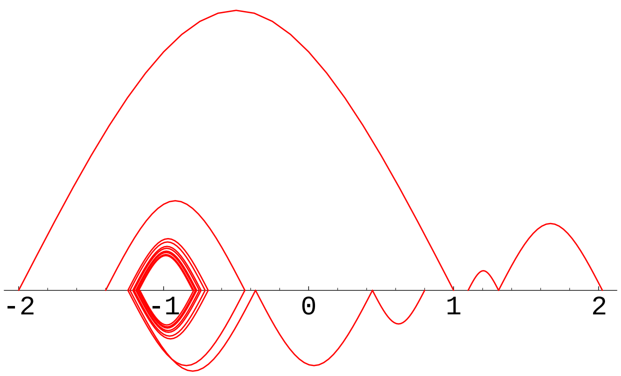
```
In[79]:= BounceDiagram[
           F-1,
           {x, -2, 2},
           {{0.8, 20},
            {1.1, 2}}]
```

In[79]:= **BounceDiagram**[
F_{-1},
{**x**, **-2**, **2**},
{{**0.8**, **20**},
{**1.1**, **2**}}]



Out[79]= - Graphics -

In[80]:= **PhaseDiagram**[
F_{-1},
{**x**, **-2**, **2**},
{{**0.8**, **10**},
{**-1.4**, **10**},
{**-2**, **3**},
{**1.1**, **2**}},
PlotRange -> **All**]

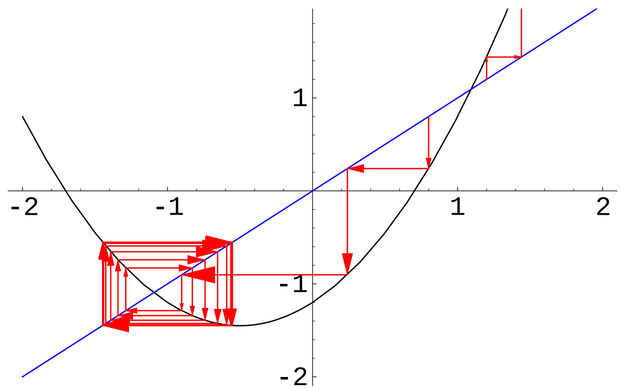


Out[80]= - Graphics -

So here our negative fixed point is attracting, but critically so. Now let's decrease further to $\lambda=-1.2$.

In[81]:= **BounceDiagram**[
$F_{-1.2}$,
{**x**, **-2**, **2**},
{{**0.8**, **20**},
{**1.2**, **2**}}]

```
In[81]:= BounceDiagram[
          F-1.2,
          {x, -2, 2},
          {{0.8, 20},
           {1.2, 2}}]
```




```
Out[83]= - Graphics -
```

```
In[84]:= PhaseDiagram[
          Nest[F-1.2, x, 2]&,
          {x, -2, 2},
          {{0.8, 20}},
          PlotRange -> All]
           {-2.2, 3},
           {1.2, 2}},
          PlotRange -> All]
```

```
Out[84]= - Graphics -
```

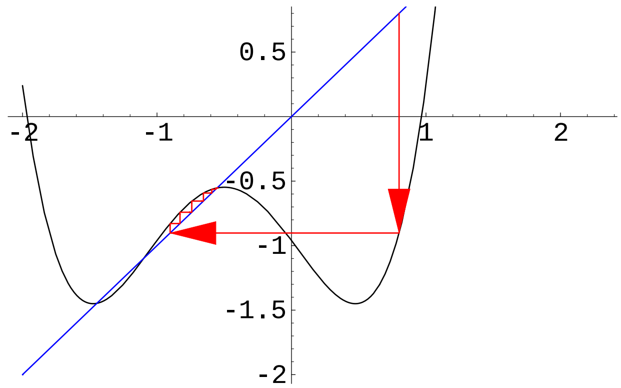
So it is a two-cycle. That's good, because it is there by our theory.

```
Out[82]= - Graphics -
```

This looks like a two−cycle. Let's repeat the analysis on $(F_{-1.2})^2$.

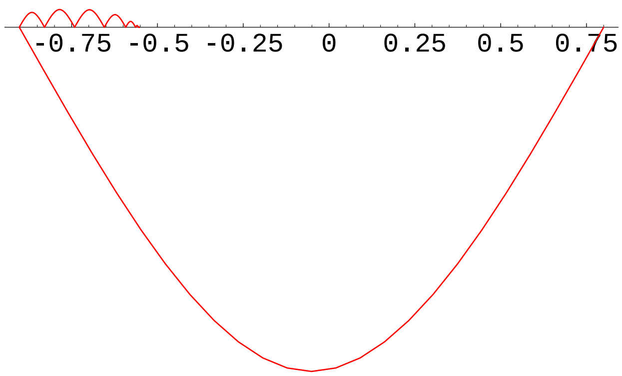
```
In[83]:= BounceDiagram[
          Nest[F-1.2, x, 2]&,
          {x, -2, 2},
          {{0.8, 20},
           {1.2, 2}}]
```

```
In[83]:= BounceDiagram[
          Nest[F-1.2, x, 2]&,
          {x, -2, 2},
          {{0.8, 20},
           {1.2, 2}}]
```




```
Out[83]= - Graphics -
```

```
In[84]:= PhaseDiagram[
          Nest[F-1.2, x, 2]&,
          {x, -2, 2},
          {{0.8, 20}},
         PlotRange -> All]
```




```
Out[84]= - Graphics -
```

So it *is* a two−cycle. That's good, because it is there by our theory.

■ Exercise 1c: $G_\mu(x) = \mu x + x^3$

■ Calculus

```
In[85]:= G[μ_, x_] := μ x + x^3
```

```
In[86]:= Gμ_[x_] := G[μ, x]
```

```
In[93]:= Solve[Gμ[Gμ[x]] == x, x]
```

```
In[87]:= Show[Plot[{G-0.8[x], x, G-1[x], G-1.2[x]},
            {x, -2, 2},
            DisplayFunction -> Identity],
          Graphics[{
            Blue,
            Line[{{-2, -2}, {2, 2}}]}],
          DisplayFunction -> $DisplayFunction]
```

Out[93]= $\left\{\{x\to 0\},\ \{x\to -\sqrt{-1-\mu}\},\ \{x\to \sqrt{-1-\mu}\},\ \{x\to -\sqrt{1-\mu}\},\ \{x\to \sqrt{1-\mu}\},\ \left\{x\to -\frac{\sqrt{-\mu-\sqrt{-4+\mu^2}}}{\sqrt{2}}\right\},\ \left\{x\to \frac{\sqrt{-\mu-\sqrt{-4+\mu^2}}}{\sqrt{2}}\right\},\ \left\{x\to -\frac{\sqrt{-\mu+\sqrt{-4+\mu^2}}}{\sqrt{2}}\right\},\ \left\{x\to \frac{\sqrt{-\mu+\sqrt{-4+\mu^2}}}{\sqrt{2}}\right\}\right\}$

The ones we care about are the ones which are real near μ=−1.

```
In[94]:= Select[%, RealQ[x /. # /. {μ -> -1}]&]
```

Out[94]= $\{\{x\to 0\},\ \{x\to -\sqrt{-1-\mu}\},\ \{x\to \sqrt{-1-\mu}\},\ \{x\to -\sqrt{1-\mu}\},\ \{x\to \sqrt{1-\mu}\}\}$

And of these we want the ones which are not already fixed points.

Out[87]= - Graphics -

It's the fixed point at 0 that we are concerned about.

```
In[88]:= D[Gμ[x], x]
         % /. {x -> 0}
```

Out[88]= $3\,x^2+\mu$

Out[89]= μ

Ah, so as μ passes through −1, we might have a bifurcation.

```
In[90]:= Φa[G][-1, 0]
```

Out[90]= −1

And since we're in the orthogonal situation, it could be period−doubling.

```
In[91]:= {Φe[G][-1, 0], Φf[G][-1, 0]}
```

Out[91]= {1, −12}

Indeed it is going to be a period−doubling one.

■ Algebra

```
In[92]:= RealQ[z_] := (Im[z] == 0)
```

```
General::spell1 :
 Possible spelling error: new symbol name "RealQ" is similar to existing symbol "Real".
```

In[93]:= `Solve[Gμ[Gμ[x]] == x, x]`

Out[93]= $\left\{\{x\to 0\},\{x\to -\sqrt{-1-\mu}\},\{x\to \sqrt{-1-\mu}\},\{x\to -\sqrt{1-\mu}\},\{x\to \sqrt{1-\mu}\},\left\{x\to -\frac{\sqrt{-\mu-\sqrt{-4+\mu^2}}}{\sqrt{2}}\right\},\left\{x\to \frac{\sqrt{-\mu-\sqrt{-4+\mu^2}}}{\sqrt{2}}\right\},\left\{x\to -\frac{\sqrt{-\mu+\sqrt{-4+\mu^2}}}{\sqrt{2}}\right\},\left\{x\to \frac{\sqrt{-\mu+\sqrt{-4+\mu^2}}}{\sqrt{2}}\right\}\right\}$

The ones we care about are the ones which are real near $\mu = -1$.

In[94]:= `Select[%, RealQ[x /. # /. {μ -> -1}]&]`

Out[94]= $\{\{x\to 0\},\{x\to -\sqrt{-1-\mu}\},\{x\to \sqrt{-1-\mu}\},\{x\to -\sqrt{1-\mu}\},\{x\to \sqrt{1-\mu}\}\}$

And of these we want the ones which are not already fixed points.

In[95]:=
```
Select[%,
  ((Simplify[(G[μ, x] /. #)]) =!= (x /. #))&]
```

Out[95]= $\{\{x\to -\sqrt{-1-\mu}\},\{x\to \sqrt{-1-\mu}\}\}$

Out[109]= - Graphics -

In[96]:= `D[Gμ[Gμ[x]], x] /. %`

In[100]:= `Sqrt[1.8]`

Out[96]= $\left\{\mu\,(3\,(-1-\mu)+\mu)+3\,(3\,(-1-\mu)+\mu)\,\left(-(-1-\mu)^{3/2}-\sqrt{-1-\mu}\,\mu\right)^2,\ \mu\,(3\,(-1-\mu)+\mu)+3\,(3\,(-1-\mu)+\mu)\,\left((-1-\mu)^{3/2}+\sqrt{-1-\mu}\,\mu\right)^2\right\}$

Out[100]= 1.34164

In[97]:= `Simplify[%]`

Out[97]= $\{(3+2\mu)^2,(3+2\mu)^2\}$

In[98]:= `Solve[%[[1]] == 1, μ]`

Out[98]= $\{\{\mu\to -2\},\{\mu\to -1\}\}$

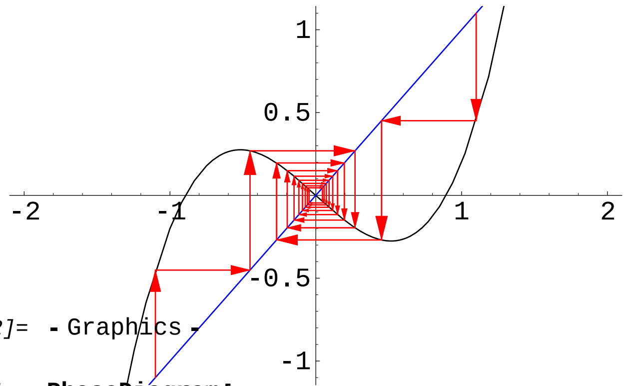
So it the two−cycle will be stable as long as $-2<\mu<-1$.

■ Graphical Analysis

In[99]:=
```
BounceDiagram[
  G-0.8,
  {x, -2, 2},
  {{-1.1, 10},
   {1.1, 10}}]
```

■ Graphical Analysis

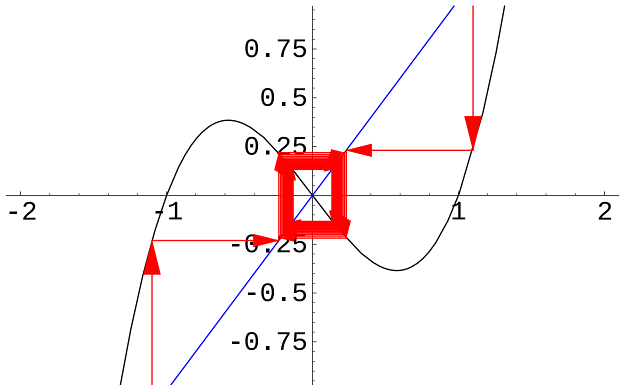
```
In[99]:= BounceDiagram[
          G-0.8,
          {x, -2, 2},
          {{-1.1, 10},
           {1.1, 10}}]
```




```
Out[99]= - Graphics -

In[100]:= Sqrt[1.8]

Out[100]= 1.34164

In[101]:= PhaseDiagram[
           G-0.8,
           {x, -2, 2},
          {{1.1, 10}, {-1.1, 10}, {1.35, 3}, {-1.35, 3}},
          PlotRange -> All]

In[102]:= BounceDiagram[
           G-1,
           {x, -2, 2},
           {{-1.1, 20},
            {1.1, 20}}]

Out[102]= - Graphics -

In[103]:= PhaseDiagram[
           G-1,
           {x, -2, 2},
           {{-1.1, 20},
            {1.1, 20},
           {-1.5, 2}, {1.5, 2}},
          PlotRange -> All]

Out[103]= - Graphics -

Out[101]= - Graphics -
```

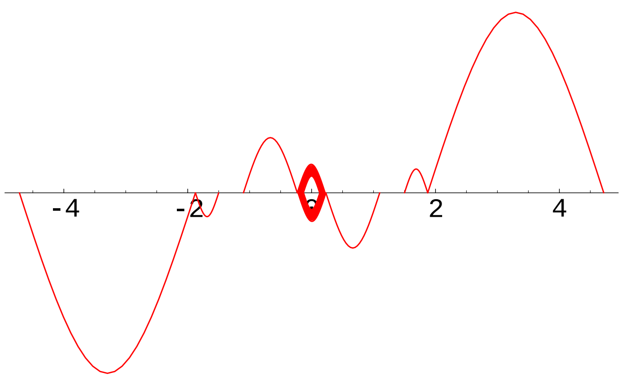
So the fixed point at zero is attracting. We also have the repelling fixed points at +/$-\sqrt{1.8}$. Let's decrease μ to the critical parameter -1.

```
In[102]:= BounceDiagram[
            G-1,
            {x, -2, 2},
            {{-1.1, 20},
             {1.1, 20}}]
```




```
Out[102]= - Graphics -
```

```
In[103]:= PhaseDiagram[
            G-1,
            {x, -2, 2},
            {{-1.1, 20},
             {1.1, 20},
            {-1.5, 2}, {1.5, 2}},
           PlotRange -> All]
```

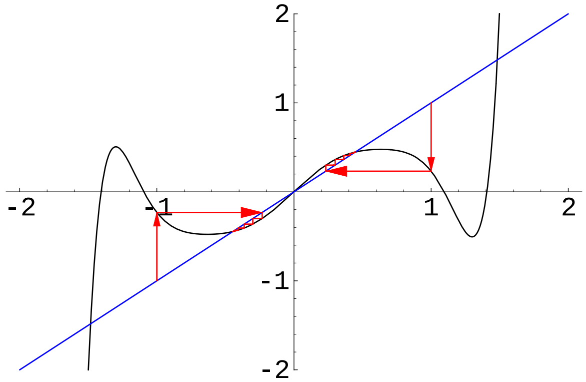



```
Out[103]= - Graphics -
```

This looks like a fixed point which is critically attracting. Let's decrease μ further and see the two−cycle.

```
In[104]:= BounceDiagram[
            G-1.2,
            {x, -2, 2},
            {{-1, 10},
             {1, 10}}]
```

```
In[106]:= BounceDiagram[
            Nest[G-1.2, #, 2]&,
            {x, -2, 2},
            {{-1, 10},
             {1, 10}}]
```




```
Out[106]= - Graphics -
```

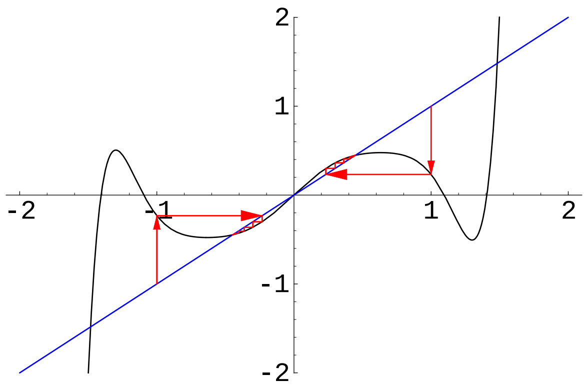
Sure enough!

```
In[105]:= PhaseDiagram[
            G-1.2,
            {x, -2, 2},
            {{-1, 10},
             {1, 10},
            {-1.5, 2}, {1.5, 2}},
          PlotRange -> All]
```

```
Out[105]= - Graphics -
```

Looks like a two−cycle. Just for good measure, we'll do it on $G_{\mu}{}^{2}$:

```
In[106]:= BounceDiagram[
            Nest[G-1.2, #, 2]&,
            {x, -2, 2},
            {{-1, 10},
             {1, 10}}]
```




```
Out[106]= - Graphics -
```

Sure enough!

■ Exercise 1g: $F_c(x) = x^3 + c$

■ Calculus

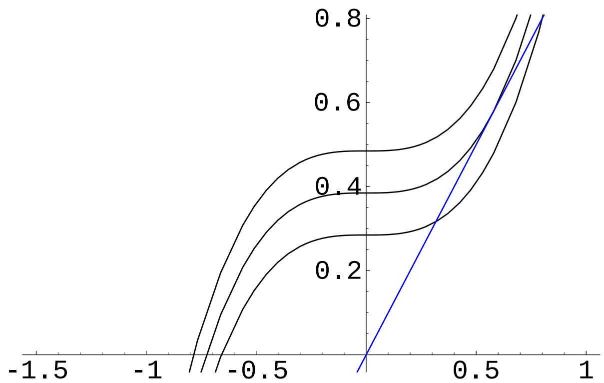
```
In[107]:= F[c_, x_] := c + x^3
```

In[108]:= $F_{c\_}$[x_] := F[c, x]

In[109]:= c0 = $\frac{2}{3\sqrt{3}}$

Out[109]= $\frac{2}{3\sqrt{3}}$

```
In[110]:= Show[Plot[{F_{c0-0.1}[x], F_{c0}[x], F_{c0+0.1}[x]},
             {x, -1.5, 1},
             DisplayFunction -> Identity],
           Graphics[{
             Blue,
             Line[{{-1.5, -1.5}, {1, 1}}]}],
           DisplayFunction -> $DisplayFunction]
```



Out[110]= - Graphics -

It looks like a saddle−node bifurcation at c_0.

```
In[111]:= Solve[F_{c0}[x] == x, x]
```

Out[111]= $\left\{\left\{x \to -\frac{2}{\sqrt{3}}\right\}, \left\{x \to \frac{1}{\sqrt{3}}\right\}, \left\{x \to \frac{1}{\sqrt{3}}\right\}\right\}$

The double−root is a good indication that we're at a saddle−node.

```
In[112]:= x0 = x /. %[[3]]
```

Out[112]= $\frac{1}{\sqrt{3}}$

```
In[113]:= {Φa[F][c0, x0], Φb[F][c0, x0], Φc[F][c0, x0]}
```

Out[113]= $\{1, 2\sqrt{3}, 1\}$

So we do have a saddle−node bifurcation by Proposition 2.2.1.

```
In[114]:= Φa[G][-1, 0]
```

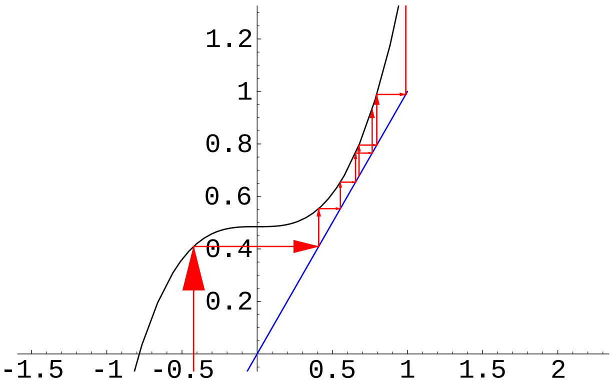
Out[114]= -1

■ Algebra

The algebra for this problem involves solving the cubic, which is quite difficult. Since we have no doubt of the existence of the saddle−node, we will proceed to the pictures to visualize it.

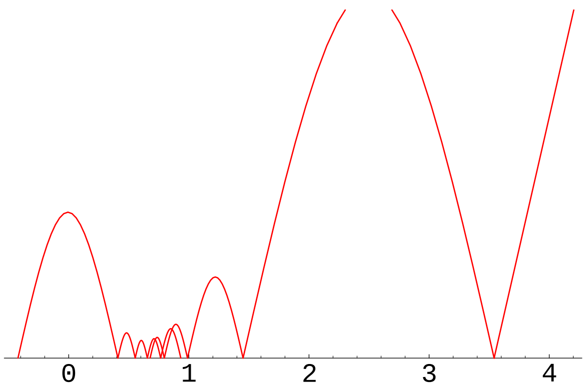
■ Graphical Analysis

In[115]:=
```
BounceDiagram[
  F_{c0+0.1},
  {x, -1.5, 1},
  {{x0 - 1, 5},
   {x0 + 0.1, 5}}]
```



Out[115]= - Graphics -

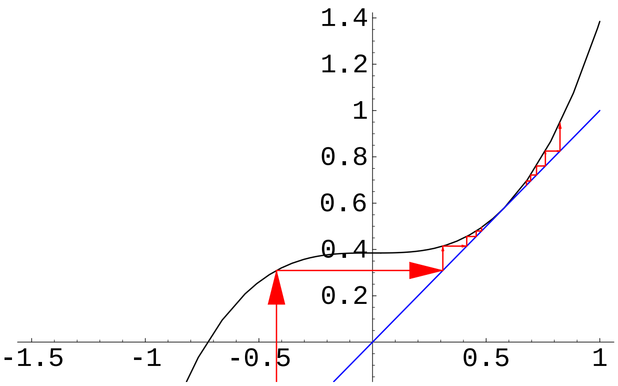
In[116]:=
```
PhaseDiagram[
  F_{c0+0.1},
  {x, -1.5, 1},
  {{x0 - 1, 5},
   {x0 + 0.1, 5}}]
```



Out[116]= - Graphics -

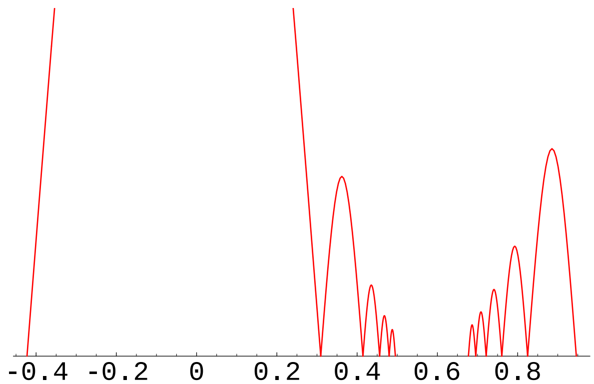
Sothere is no fixed point in the region we're focusing on. We do have a repelling fixed point which is negative, but we're not worried about that one. Let's decrease c to the critical parameter c_0.

```
In[117]:= BounceDiagram[
            Fc0,
            {x, -1.5, 1},
            {{x0 - 1, 5},
             {x0 + 0.1, 5}}]
```




```
Out[117]= - Graphics -
```

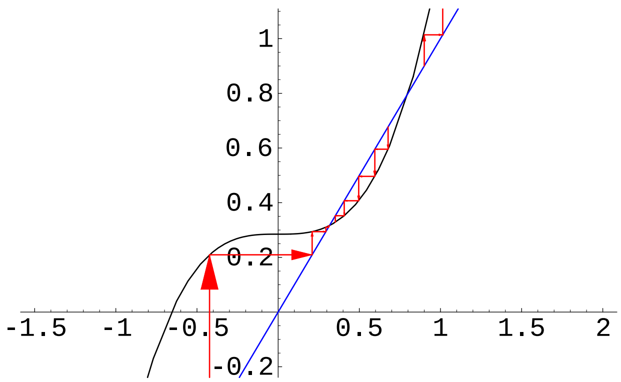
```
In[118]:= PhaseDiagram[
            Fc0,
            {x, -1.5, 1},
            {{x0 - 1., 5},
             {x0 + 0.1, 5}}]
```




```
Out[118]= - Graphics -
```

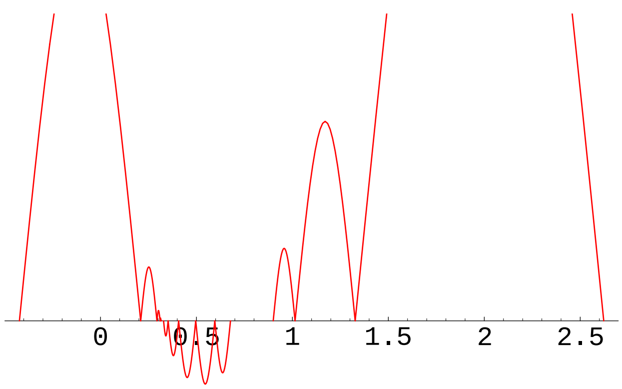
So we have a fixed point, but it is attracting on the left and repelling on the right. Let's decrease *c* just a bit further.

```
In[119]:= BounceDiagram[
            Fc0-0.1,
            {x, -1.5, 2},
            {{x0 - 1, 5},
             {x0 + 0.1, 5},
            {0.9, 3}}]
```




```
Out[119]= - Graphics -
```

```
In[120]:= PhaseDiagram[
            Fc0-0.1,
            {x, -1.5, 2},
            {{x0 - 1, 5},
             {x0 + 0.1, 5},
            {0.9, 3}}]
```




```
Out[120]= - Graphics -
```

So we now have the "birth" of a repelling and an attracting fixed point.

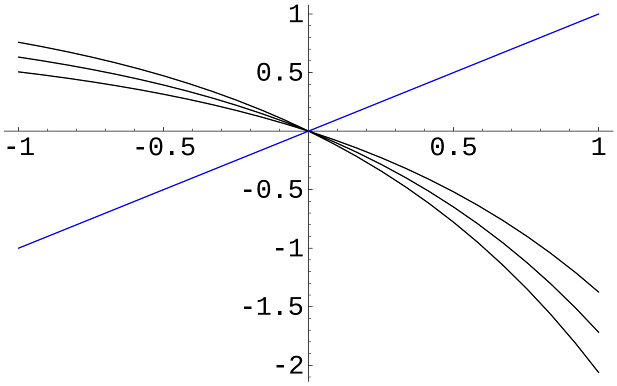
■ Exercise 1h: $e_\lambda(x) = \lambda(\exp(x) - 1)$

■ Calculus

```
In[121]:= e[λ_, x_] := λ (Exp[x] - 1)
```

```
In[122]:= eλ_[x_] := e[λ, x]
```

```
In[123]:= Show[Plot[{e_{-1.2}[x], e_{-1}[x], e_{-0.8}[x]},
              {x, -1, 1},
              DisplayFunction -> Identity],
            Graphics[{
              Blue,
              Line[{{-1, -1}, {1, 1}}]}],
            DisplayFunction -> $DisplayFunction]
```




```
Out[123]= - Graphics -
```

So the only fixed point in sight for λ near -1 is the fixed point 0. As λ approaches -1, it looks like we're in the period−doubling situation.

```
In[124]:= {Φa[e][-1, 0], Φe[e][-1, 0], Φf[e][-1, 0]}
```

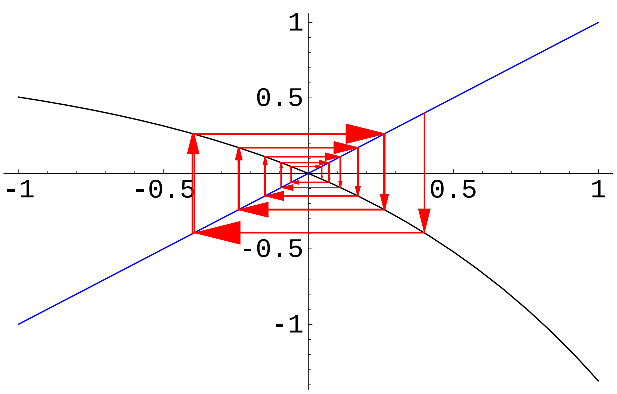
Out[124]= $\left\{-1, \frac{3}{2}, -1\right\}$

Indeed we are.

■ Graphical analysis

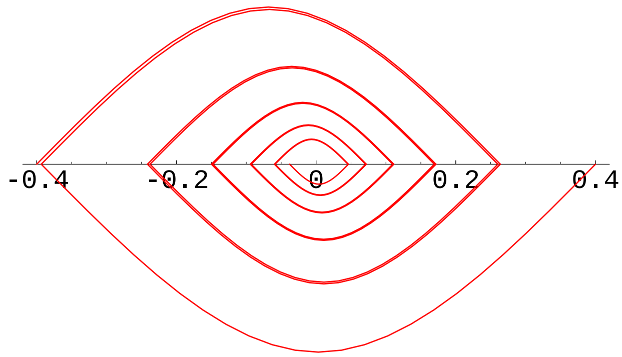
Solving $e_\lambda(x) = x$ is transcendental, so we won't attempt it. Instead, we'll just graph it.

```
In[125]:= BounceDiagram[e_{-0.8},
            {x, -1, 1},
            {{0.4, 10},
            {-0.4, 10}}]
```




```
Out[125]= - Graphics -
```

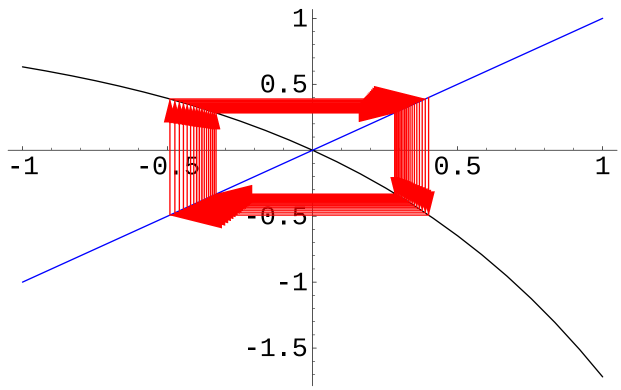
```
In[126]:= PhaseDiagram[e_{-0.8},
           {x, -1, 1},
           {{0.4, 10},
           {-0.4, 10}}]
```




```
Out[126]= - Graphics -
```

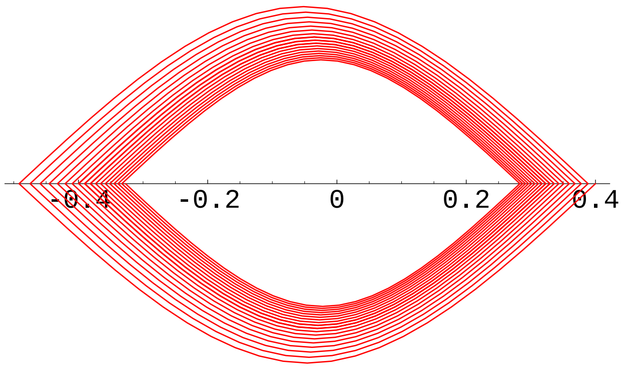
So the fixed point at 0 is attracting for λ slightly bigger than -1. Let's set λ equal to its critical value -1.

```
In[127]:= BounceDiagram[e_{-1},
           {x, -1, 1},
           {{0.4, 20},
           {-0.4, 20}}]
```




```
Out[127]= - Graphics -
```

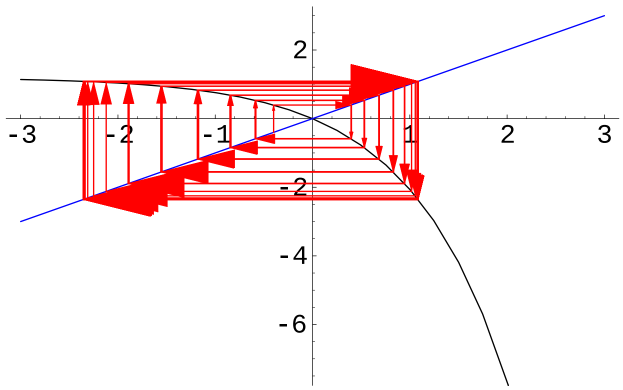
In[128]:= **PhaseDiagram[e_{-1},**
{x, -1, 1},
{{0.4, 20},
{-0.4, 20}}]



Out[128]= - Graphics -

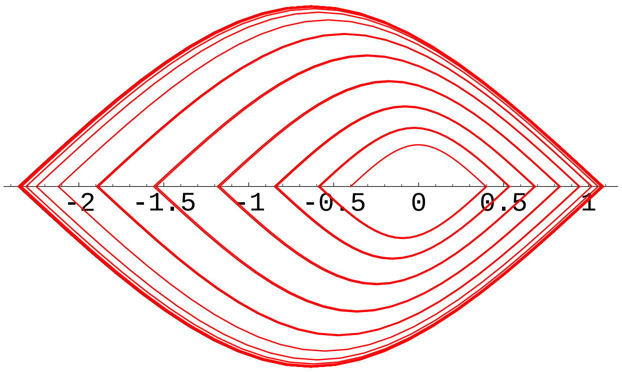
This is attracting, but critically so. Finally, for λ smaller than -1.

In[129]:= **BounceDiagram[$e_{-1.2}$,**
{x, -3, 3},
{{0.4, 10},
{-0.4, 40}}]



Out[129]= - Graphics -

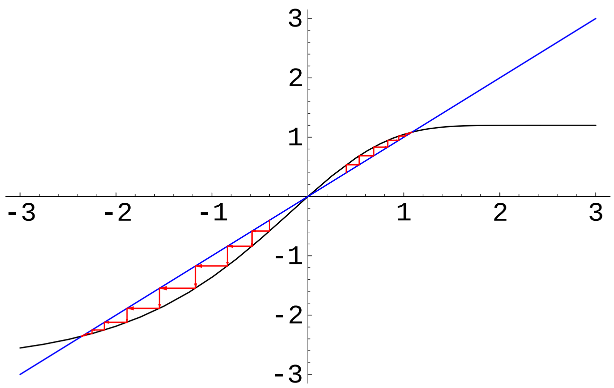
```
In[130]:= PhaseDiagram[e₋1.2,
            {x, -3, 3},
            {{0.4, 10},
            {-0.4, 40}},
           PlotRange -> All]
```




```
Out[130]= - Graphics -
```

It looks like a two−cycle.

```
In[131]:= BounceDiagram[Nest[e₋1.2, #, 2]&,
            {x, -3, 3},
            {{0.4, 10},
            {-0.4, 40}}]
```




```
Out[131]= - Graphics -
```

Indeed it is.

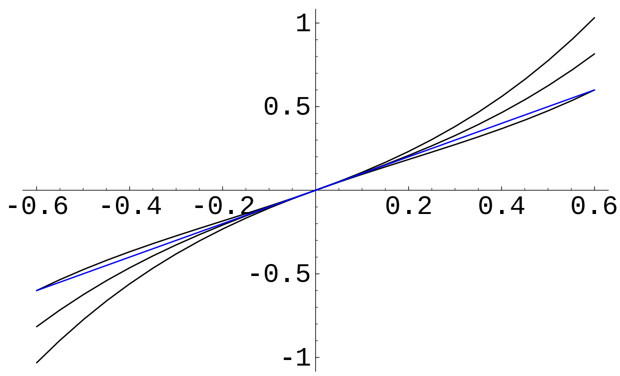
■ Exercise 1k: $F_c(x) = x + c\,x^2 + x^3$

```
In[132]:= F[c_, x_] := x + c x^2 + x^3
```

```
In[133]:= F_c_[x_] := F[c, x]
```

■ Calculus

```
In[134]:= Plot[{F₋₀.₆[x], F₀[x], F₀.₆[x], x},
            {x, -0.6, 0.6},
            PlotStyle -> {{}, {}, {}, {Blue}}]
```




```
Out[134]= - Graphics -
```

```
In[135]:= Φa[F][0, 0]
```

```
Out[135]= 1
```

So we could be in the fold situation.

```
In[136]:= {Φb[F][0, 0], Φc[F][0, 0]}
```

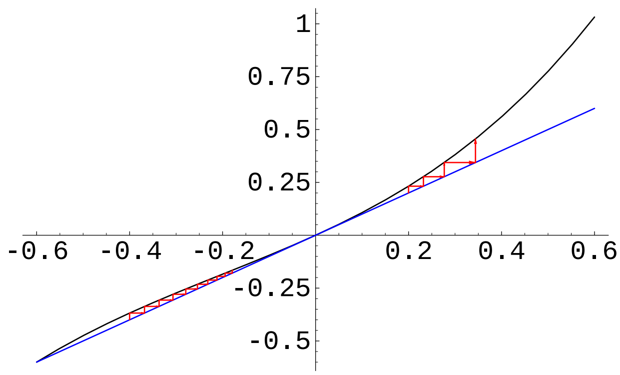
```
Out[136]= {0, 0}
```

But our higher derivative tests fail!

■ Graphical Analysis

Let’s see how the fold bifurcation manages to fail.

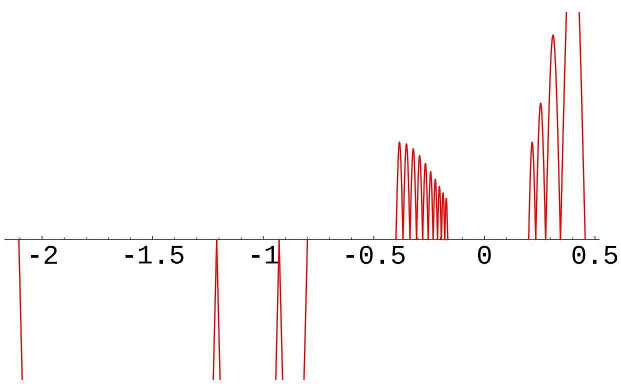
```
In[137]:= BounceDiagram[F0.6,
            {x, -0.6, 0.6},
            {{0.2, 4},
             {-0.4, 10}}]
```




```
Out[137]= - Graphics -
```

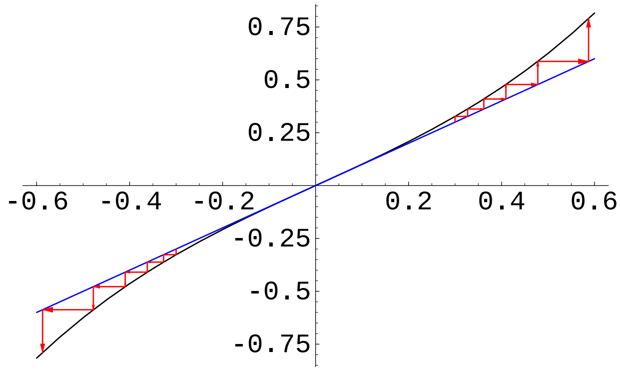
So we have a negative fixed point which is repelling, and a fixed point at zero which is attracting on the left, but repelling on the right.

```
In[138]:= PhaseDiagram[F0.6,
            {x, -0.6, 0.6},
            {{0.2, 4},
             {-0.4, 10},
            {-0.8, 3}}]
```



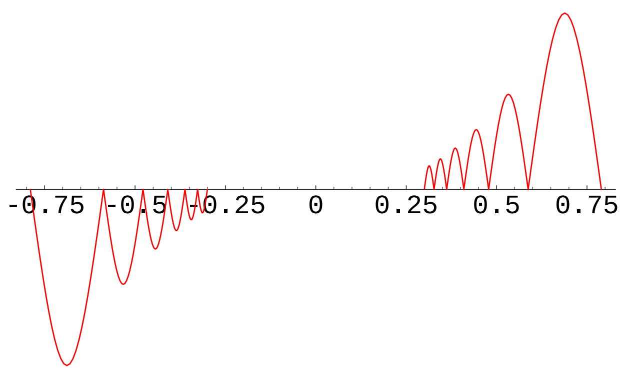

```
Out[138]= - Graphics -
```

```
In[139]:= BounceDiagram[F0,
            {x, -0.6, 0.6},
            {{0.3, 6},
             {-0.3, 6}}]
```




```
Out[139]= - Graphics -
```

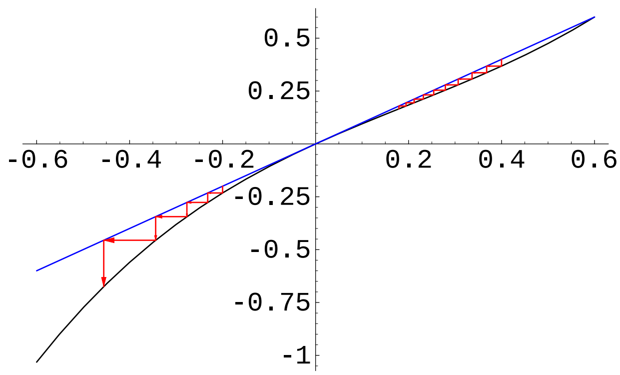
```
In[140]:= PhaseDiagram[F0,
            {x, -0.6, 0.6},
            {{0.3, 6},
             {-0.3, 6}}]
```




```
Out[140]= - Graphics -
```

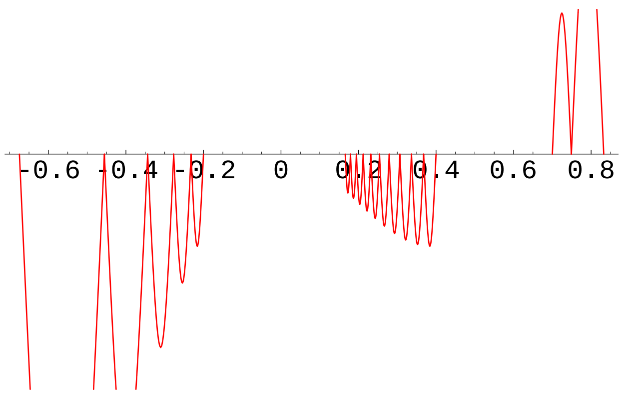
Here we have a single fixed point which is repelling, but slowly.

```
In[141]:= BounceDiagram[F-0.6,
            {x, -0.6, 0.6},
            {{0.4, 10},
             {-0.2, 5}}]
```




```
Out[141]= - Graphics -
```

```
In[142]:= PhaseDiagram[F-0.6,
            {x, -0.6, 0.6},
            {{0.4, 10},
             {-0.2, 5}, {0.7, 2}}]
```




```
Out[142]= - Graphics -
```

So here our fixed point at zero is repelling on the left but attracting on the right. There is a second fixed point, which is positive, and is repelling.

■ Algebra

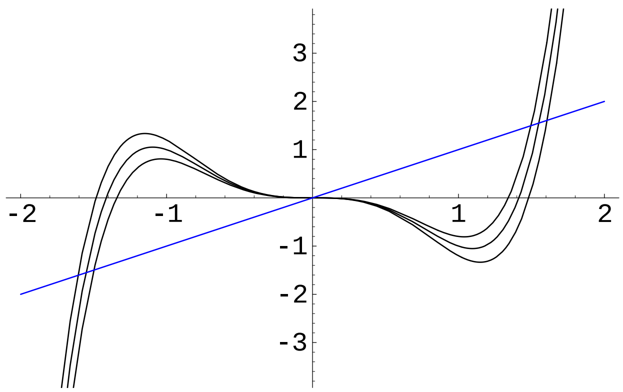
Why did this happen? We can solve the equations explicitly here:

```
In[143]:= Solve[Fc[x] == x, x]

Out[143]= {{x → 0}, {x → 0}, {x → -c}}

In[144]:= D[Fc[x], x] /. %

Out[144]= {1, 1, 1 + c^2}
```

There is a double−root at 0, which often means we get some kind of degeneracy. Note that this fixed point is always nonhyperbolic. There is a second fixed point, which as c moves past the critical parameter value 0, actually moves *through* the other fixed point, and which is repelling except when c=0, at which point it is only nonhyperbolic. No wonder things got so screwed up!

■ Exercise 16: $F_\lambda(x) = x^5 - \lambda x^3$

```
In[145]:= F[λ_, x_] := x^5 - λ x^3

In[146]:= Fλ_[x_] := F[λ, x]

In[147]:= Plot[{F1.8[x], F2[x], F2.2[x], x},
            {x, -2, 2},
           PlotStyle -> {{}, {}, {}, {Blue}}]
```



```
Out[147]= - Graphics -
```

So we have a fixed point at zero matter what λ is. It also looks like we get two other fixed points, symmetric about 0 since F_λ is odd, which are repelling.

```
In[148]:=

In[149]:= Solve[F[λ, x] == x, x]
```

Out[149]= $\left\{\{x \to 0\}, \left\{x \to -\frac{\sqrt{\lambda - \sqrt{4+\lambda^2}}}{\sqrt{2}}\right\}, \left\{x \to \frac{\sqrt{\lambda - \sqrt{4+\lambda^2}}}{\sqrt{2}}\right\}, \left\{x \to -\frac{\sqrt{\lambda + \sqrt{4+\lambda^2}}}{\sqrt{2}}\right\}, \left\{x \to \frac{\sqrt{\lambda + \sqrt{4+\lambda^2}}}{\sqrt{2}}\right\}\right\}$

```
In[150]:= Select[%, RealQ[x /. # /. {λ -> 2}]&]
```

Out[150]= $\left\{\{x \to 0\}, \left\{x \to -\frac{\sqrt{\lambda + \sqrt{4+\lambda^2}}}{\sqrt{2}}\right\}, \left\{x \to \frac{\sqrt{\lambda + \sqrt{4+\lambda^2}}}{\sqrt{2}}\right\}\right\}$

These are the ones which are real for λ near 2.

```
In[151]:= D[Fλ[x], x] /. %
```

Out[151]= $\left\{0, -\frac{3}{2}\lambda\left(\lambda+\sqrt{4+\lambda^2}\right)+\frac{5}{4}\left(\lambda+\sqrt{4+\lambda^2}\right)^2, -\frac{3}{2}\lambda\left(\lambda+\sqrt{4+\lambda^2}\right)+\frac{5}{4}\left(\lambda+\sqrt{4+\lambda^2}\right)^2\right\}$