

MATH 118: SPRING 1999
SOLUTIONS TO PROBLEM SET 2

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RATES OF CONVERGENCE

See attached *Mathematica* output

A “NEUTRAL” FIXED POINT THEOREM

Exercise 5.5. Suppose that F has a “neutral” fixed point at x_0 with $F'(x_0) = 1$. Suppose also that $F''(x_0) > 0$. What can you say about x_0 ?

Answer. By a change of coordinates (a simple application of the Implicit Function Theorem), we can assume $F(x) = x + \frac{k}{2}x^2$, where $k = F''(0)$. Then on the regions where F is increasing and concave up, iterates of F must increase yet stay between the curve and the diagonal (see Figure 1). Thus F is attracting on the left, but repelling on the right. □

Exercise 5.6. Repeat Exercise 5, but this time assume that $F''(x_0) < 0$.

Answer. Apply Exercise 5 to the function $G(x) = -F(-x)$, we see that the fixed point must be attracting from the right but repelling from the left. □

Date: March 5, 1999.

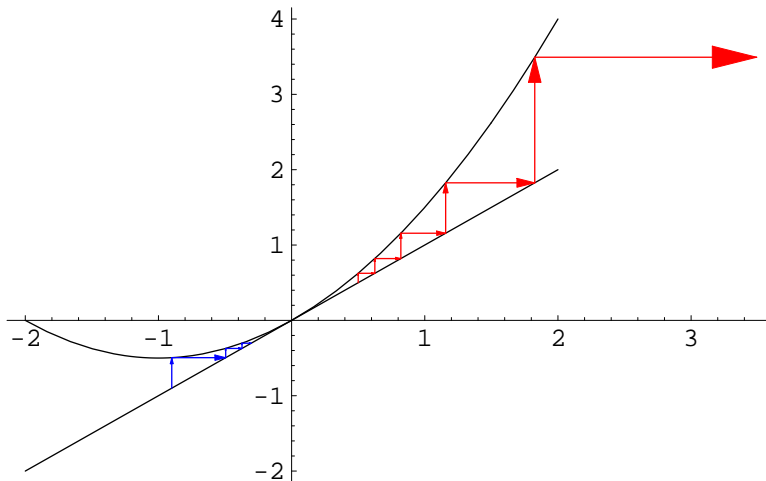


FIGURE 1. A function of the kind described in Exercise 5.5. Notice the fixed point 0 is attracting on the left, but repelling on the right.

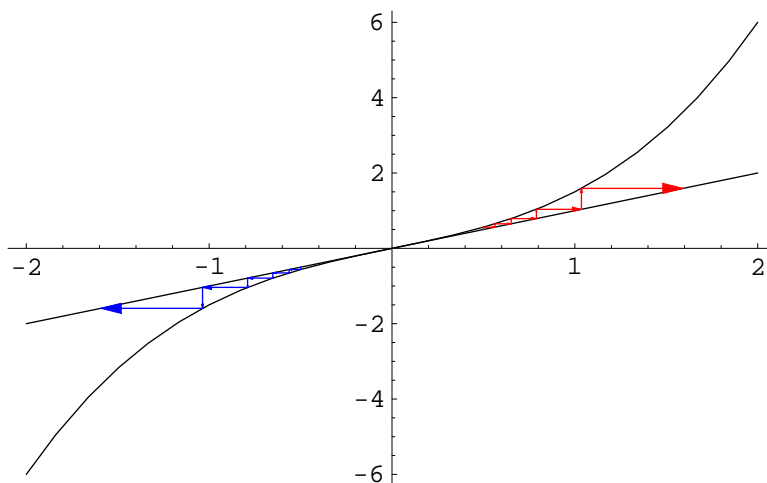


FIGURE 2. A function of the kind described in Exercise 5.7. Notice the fixed point 0 is repelling.

Exercise 5.7. Suppose that F has a “neutral” fixed point at x_0 with $F'(x_0) = 1$ and $F''(x_0) = 0$. Suppose also that $F^{(3)}(x_0) > 0$. Use graphical analysis and the concavity of the graph of F near x_0 to show that x_0 is repelling.

Answer. Because the curve lies above the diagonal and is concave up for positive x , we have that x_0 repels on the right. But since the figure is rotationally symmetric, we have repulsion on the left, too. See Figure 2. \square

Exercise 5.8. Repeat Exercise 7, but this time assume that $F^{(3)}(x_0) < 0$. Show that x_0 is attracting.

Answer. Clear. \square

Exercise 5.9. Combine the results of Exercises 5–8 to state a “Neutral” Fixed Point Theorem.

Answer. Let F have a fixed point x_0 , with $F'(x_0) = 1$. Then if $F''(x_0) > 0$, (respectively, < 0), x_0 is attracting on the left (respectively, right) and repelling on the right (respectively, left). If $F''(x_0) = 0$ and $F^{(3)}(x_0) > 0$ (respectively, < 0), then x_0 is repelling (respectively, attracting). \square

MORE ON NEWTON’S METHOD

Exercise 13.2. What happens when Newton’s method is applied to $F(x) = \sqrt{x}$?

Answer. Notice that $N_F(x) = -x$. So we’re in big trouble even after one iteration, because F is not defined for negative x . If we extend F by defining $F(x) = \sqrt{|x|}$ or $F(x) = \text{sgn}(x)\sqrt{\text{abs}x}$, we still get the same function for N_F . Thus every nonzero point is on a (truly neutral) two-cycle of N_F . \square

Exercise 13.3. Find all fixed points for the associated Newton iteration function for $F(x) = x/(x-1)^n$ when $n = 1, 2, 3, \dots$. Which are attracting and which are repelling?

Answer. According to Exercise 13.5, the attracting fixed points of N_F are the zeros of the numerator, namely 0, and this is a superattracting fixed point. The other fixed points are the zeros of the denominators, namely 1, and this fixed point is repelling. \square

Exercise 13.4. Consider the Newton iteration function for $F(x) = \sec x$. What are the fixed points for N_{\sec} ? Does this contradict the Newton Fixed Point Theorem?

Answer. $N_{\sec}(x) = x - \cot x$. The fixed points of N_{\sec} are at the zeros of $\cot x$, which are at $x = \frac{(2k+1)\pi}{2}$. The secant function has a pole at these points, but this does not violate the Newton Fixed Point Theorem, since these points are not in the domain of definition of \sec . In fact, \sec has no real roots. \square

Exercise 13.5. Suppose that $P(x)$ and $Q(x)$ are polynomials and let $F(x) = P(x)/Q(x)$. What can be said about the fixed points of the associated Newton function for F ? Which fixed points are attracting and which are repelling?

Answer. Notice that

$$N_F(x) = x - \frac{P/Q}{\frac{Q P' - P Q'}{Q^2}} = x - \frac{PQ}{P'Q - PQ'},$$

and

$$(1) \quad N'_F = 1 - \frac{(P'Q - PQ')(P'Q + PQ') - PQ(P''Q - PQ'')}{(P'Q - PQ')^2}.$$

So the fixed points of N_F are clearly those places where $P = 0$ or $Q = 0$. (Note that if they had a common root, we can cancel them out of both.) Evaluating (1) at the points where $P = 0$ gives $N'_F = 0$, i.e., a superattracting fixed point. On the other hand, if $Q = 0$, we see that $N'_F = 2$, i.e., a repelling fixed point. \square

Exercise 13.8. Consider the function $G(x) = x^4 - x^2 - 11/36$.

- Compute the inflection points of G . Show that they are critical points for the associated Newton function.
- Prove that these two points lie on a two-cycle.
- What can you say about the convergence of Newton's method for this function?

Proof. a. The inflection points are found at $p_{\pm} = \pm \frac{1}{\sqrt{6}}$.

b. Clear.

c. If we start to the left of the critical points of G , we will land at the negative root. Similarly with the positive root. But if we start in the middle, we will converge to the attracting two-cycle $\left\{ \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\}$. See Figure 3. \square

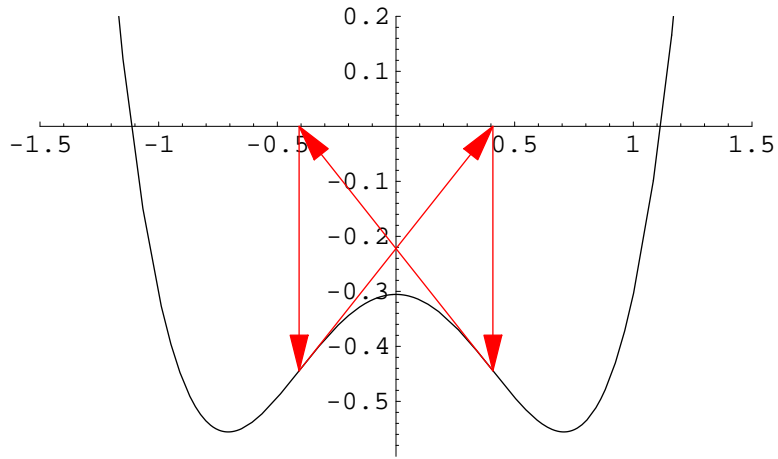


FIGURE 3. The attracting two-cycle in Newton's method for $G(x) = x^4 - x^2 - 11/36$.