MATH 118: SPRING 1999
PROBLEM SET 1 SOLUTIONS

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Exercises from Robinson:

Exercise 2.3. *Prove Theorem 2.2: Assume* $f\colon \mathbb{R} \to \mathbb{R}$ *is a* C^1 *function. Assume that* p *is a periodic point of period* n *with* $|(f^n)'(p)| > 1$. *Then* p *is repelling. Moreover, for all sufficiently small intervals* I *about* p *and* $x \in I$ *there exists a* $k = k_x$ *such that* $f^{kn}(x) \notin I$. *This says that all points near* p *go away from* p *under iterates of* f^n.

Proof. The second conclusion implies the first. We may assume $p = f(p) = 0$. Choose $\eta > 1$ and $\delta > 0$ such that $|f'(x)| > \eta$ when $|x| < \delta$. This exists because f' is continuous.

Let $U = (-\delta, \delta)$. Then for $x \in U$, we have

$$|f(x)| > \eta\,|x|\,. \tag{1}$$

We're done: Let $I \ni 0$ and suppose $I \subset U$ (this is the "sufficiently small" part). Pick $x \in I$, and note that there exists k such that $\eta^k\,|x| \notin I$. Then (1) shows that $f^k(x) \notin I$. ☐

Exercise 2.4. *Let*

$$T(x) = \begin{cases} 2x & \text{for } x \le 1/2; \\ 2 - 2x & \text{for } x \ge 1/2 \end{cases}$$

be the tent map.

(a) *Sketch the graph on* $I = [0,1]$ *of* T, T^2, *and (a representative graph of)* T^n *for* $n > 2$.
(b) *Use the graph of* T^n *to conclude that* T *has exactly* 2^n *points of period* n.
(c) *Prove that the set of all periodic points of* T *is dense in* $[0,1]$.
(d) *Find the number of distinct periodic orbits.*

Proof. (a) The graphs are shown in Figure 1. Clearly T^n has 2^{n-1} "tents" evenly spaced throughout the interval $[0,1]$.

(b) The line $y = x$ intersects the graph of T^n twice at each of these tents. Thus there are $2 \cdot 2^{n-1} = 2^n$ fixed points of T^n. These are all the points of (not necessarily prime) period n.

(c) Given $x \in [0,1]$ and $\varepsilon > 0$, there exists positive integers n and k such that

$$x - \varepsilon < \frac{k}{2^{n-1}} \le x < \frac{k+1}{2^{n-1}} < x + \varepsilon. \tag{2}$$

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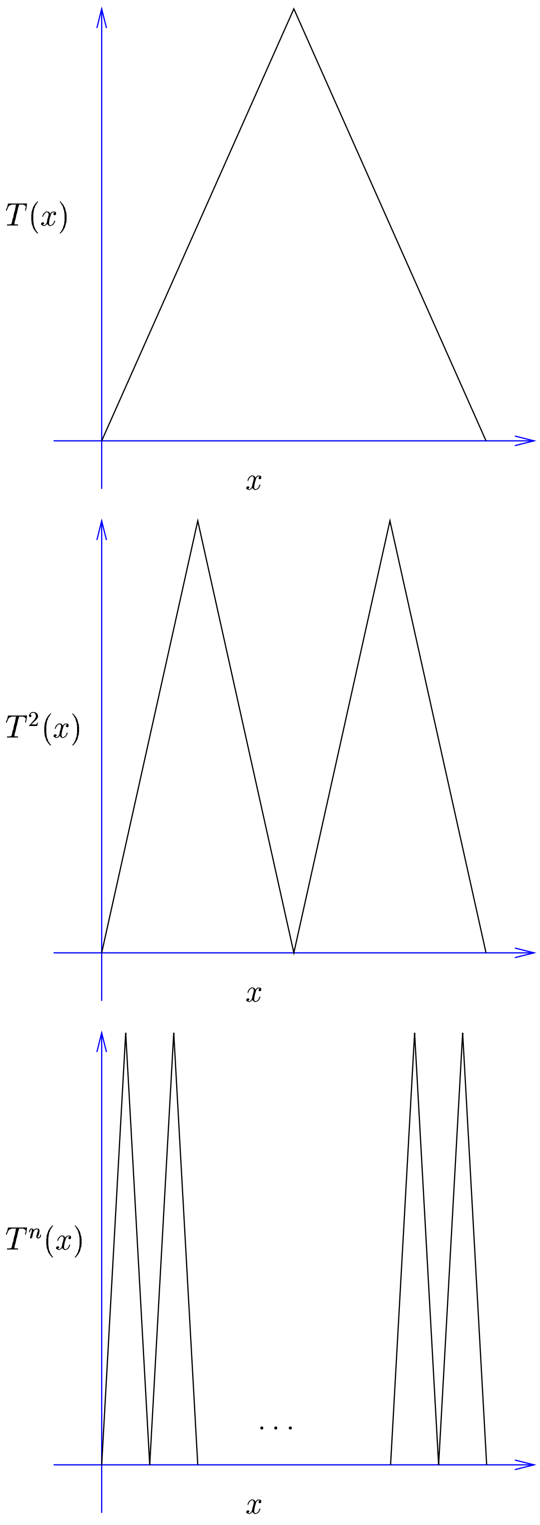


FIGURE 1. Graphs of iterates of T.

But between $\frac{k}{2^{n-1}}$ and $\frac{k+1}{2^{n-1}}$ there exists a fixed point q of T^n, i.e., a periodic point of T. Now

$$|x-q| \leq \left|x - \frac{k}{2^{n-1}}\right| + \left|\frac{k}{2^{n-1}} - q\right| \varepsilon + \frac{1}{2^{n-1}}.$$

So we go back to the beginning, and re-choose n and k such that we can replace ε by $\varepsilon/2$ in (2), and also so that $\frac{1}{2^{n-1}} < \frac{\varepsilon}{2}$.

(d) There are exactly two fixed points of T. Broken down into their orbits, they are $\{0\}$ and $\left\{\frac{2}{3}\right\}$. The single two-cycle is the orbit $\left\{\frac{2}{5}, \frac{4}{5}\right\}$.

Two find the four-cycles, we note that there are exactly $2^4 = 16$ points of period 4. Two of these are already the fixed points, and another two comprise the two-cycle. Hence there are 12 remaining 4-fold period points, and they must fall into three orbits. By inspection, we find that they are $\left\{\frac{2}{17}, \frac{4}{17}, \frac{8}{17}, \frac{16}{17}\right\}$, $\left\{\frac{6}{17}, \frac{12}{17}, \frac{10}{17}, \frac{14}{17}\right\}$, and $\left\{\frac{2}{15}, \frac{4}{15}, \frac{8}{15}, \frac{14}{15}\right\}$.

□

Exercise 2.5. *Let $F_4(x) = 4x(1-x)$ on $\mathbb{R}$.*

(a) *Make a rough sketch of the graph of $F_4^n(x)$ for $n > 2$.*

(b) *Use the graph of F_4^n to conclude that F_4^n has exactly 2^n fixed points. (These points do not necessarily have least period n but are fixed by F_4^n.)*

Proof. A graph of F_4 and (an approximation to) F_4^n is given in Figure . A continuity argument like in Exercise 2.4(b) gives the same result.

In fact, the graphs of T and F_4 look almost exactly alike except that T is not differentiable at $\frac{1}{2}$, while F_4 is. Notice that $T(x) = -|2x-1| + 1$ while $F_4(x) = -(2x-1)^2 + 1$. We claim that these maps are conjugate.

What we need is a homeomorphism $\varphi\colon [0,1] \to [0,1]$ such that $F_4 \circ \varphi = T$. That's not too hard. First, the map $\psi\colon \mathbb{R} \to \mathbb{R}$,

$$\psi(x) = \begin{cases} -\sqrt{-x} & \text{if } x < 0; \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$$

is a homeomorphism of $\mathbb{R}$. Then $\varphi(x) = \frac{1}{2}\psi(2x-1) + \frac{1}{2}$ will do the trick:

$$\begin{aligned} F_4(\varphi(x)) &= -\big(2\big(\frac{1}{2}\psi(2x-1) + \frac{1}{2}\big) - 1\big)^2 + 1 \\ &= -\big(\psi(2x-1)\big)^2 + 1 \\ &= -|2x-1| + 1 = T(x). \end{aligned}$$

□

Exercise 2.7. *Let T be the tent map defined in Exercise 2.4. Prove that $x \in [0,1]$ is periodic if and only if $x = \frac{2k}{p}$, where p is an odd integer and k is an integer.*

Proof. Let x^* be periodic with period n. Then (x^*, x^*) must be on one of the lines that make up the graph of T^n. These take the form

$$y = \pm 2^n \left(x - \frac{k}{2^{n-1}}\right).$$

Solving this gives

$$x^* = \frac{2k}{2^n \pm 1}, \tag{3}$$

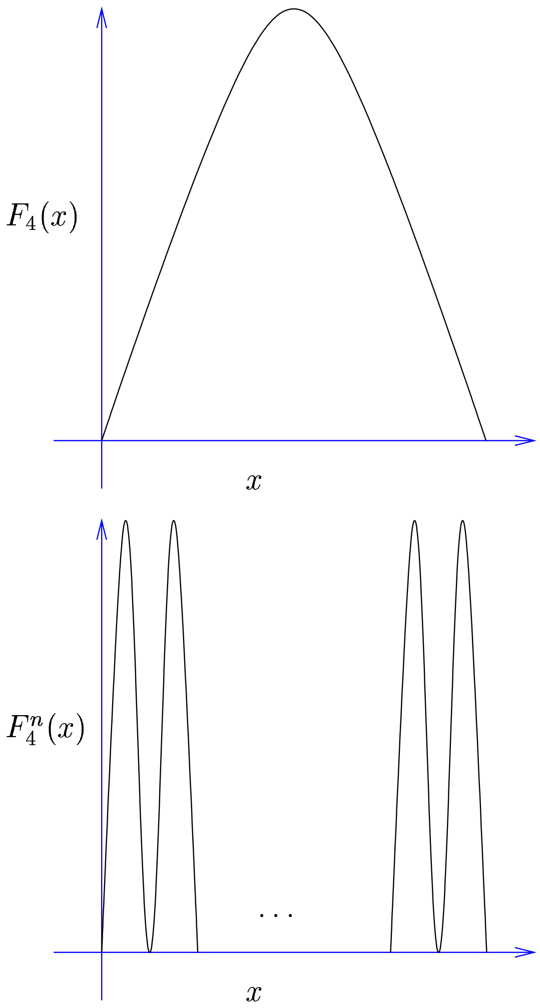


FIGURE 2. Graphs of F_4 and F_4^n

which is of the form $\frac{\text{even}}{\text{odd}}$, as desired.

Conversely, Let $x = \frac{2k}{p}$ be given, where p is odd. To show that x is periodic, it is enough to show that it can be written in the form (3). That is, we need to find integers ℓ and m such that

$$\frac{2k}{p} = \frac{2\ell}{2^m - 1}.$$

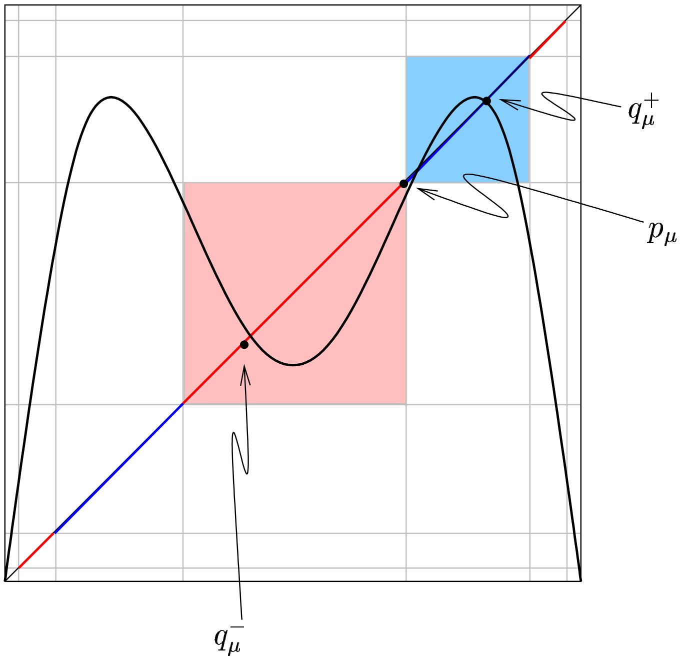
Multiplying through and canceling, we see it is enough to solve

$$2^m k = \ell p + k.$$

Now reduce the equation *modulo* p. We get

$$2^m k \equiv k \pmod{p}.$$

So it's really enough to solve $2^m k \equiv 1 \pmod{p}$ for m.



Proof.

FIGURE 3. Basins of attraction for F_μ^2, where $3 < \mu < 1 + \sqrt{6}$.

Consider the sequence $1, 2, 4, 18, 16, \ldots$ reduced *modulo* p. This sequence must have repetitions by the Pigeonhole Principle. So there exist $n > n'$ such that $2^n \equiv 2^{n'} \pmod{p}$. But since 2 is invertible *modulo* p (remember p is odd), we can get $2^{n-n'} \equiv 1 \pmod{p}$. □

Exercise 2.8. *Consider the quadratic map $F_\mu(x) = 4x(1-x)$ on $\mathbb{R}$.*

(a) *Find the points of period two and determine their stability. (Indicate for which parameter values they exist and the stability for different parameter values.)*

(b) *Let $\mu > 1$ be in the range of parameterrs for which the orbit of period two is attracting. Prove that either there exists k such that $F_\mu^k(x) = p_\mu$ (where p_μ is the fixed point), or x has the orbit of period two as its ω-limit set.*

(a) Factoring the quartic $F_\mu^2(x) = x$ (remember, we already know two factors) gives the nontrivial roots
$$q_\mu^\pm = \frac{\mu + 1 \pm \sqrt{(\mu+1)(\mu-3)}}{2\mu},$$
so there are two period-two points of F_μ as long as $\mu > 3$. Notice that this is exactly the point where both fixed points cease to be attractive. Morever,
$$(F_\mu^2)'(q_\mu^\pm) = 4 + 2\mu - \mu^2,$$
And thus $\left|(F_\mu^n)'(q_\mu^\pm)\right| < 1$ as long as $3 < \mu < 1 + \sqrt{6}$. This is the region where the two-cycle is attracting.

(b) Refer to Figure 3, although the intersections may be slightly off kilter. The pink square can be drawn around the diagonal with coordinates $[1 - p_\mu, p_\mu]^2$; this region is preserved by F_μ^2 and in fact, by an argument similar to Theorem 2.5 in Robinson, all orbits are asymptotic to the fixed point q_μ^-.

We can do a similar thing for the blue square. Then, by drawing the gray lines, we follow F_μ^2 backwards, coloring an interval on the diagonal red if it lands in the pink square, or blue if it lands in the blue one. Clearly this process will fill in the whole diagonal except for the countable endpoints of these subintervals, which map to p_μ.

$\square$