MATH 118: COMPUTER PROJECT 2
PERIOD DOUBLING AND UNIVERSALITY

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This is a computer experiment reproducing what Mitchell Feigenbaum did on an airplane trip with a pocket calculator 20 years ago. See [Fei80]. Many apologies to Michael Vinson ([Vin93]), after whose text I am fashioning this sequence of problems. It involves studying the scaling that arises in the period-doubling situation.

Let $L_\mu(x) = \mu x(1-x)$ be the logistic family of maps of the unit interval to itself.

Exercise 1. *Generate a plot of the famous "orbit diagram" for the family, which should describe the limiting behavior of L_μ versus μ. For a point $\mu \in [0,4]$, begin with a random seed x_0 and iterate L_μ, say, 100 times to let transients die down. Then plot the next 400 iterates of x, $\{(\mu, L_\mu^n(x))\}$. A useful command is `ListPlot`.*

Your code should be flexible enough to zoom in on various portions of the diagram.

You'll notice that the regions of 2^n-fold periodicity seem to be shrinking geometrically. We want to find the scaling factor of the successive bifurcations. If $\tilde{\mu}_n$ is the point just after which stable 2^n-cycles appear, then what we want is

$$\delta = \lim_{n\to\infty} \tilde{\delta}_n = \lim_{n\to\infty} \frac{\tilde{\mu}_n - \tilde{\mu}_{n-1}}{\tilde{\mu}_{n+1} - \tilde{\mu}_n}$$

A little quicker way to do this is to find not the parameter values μ where the system makes a transition from 2^n-cycles to 2^{n+1} cycles, but rather the values of μ for which *superstable* 2^n-cycles exist.

Exercise 2. *Give two reasons why finding superstable cycles is quicker and easier for a computer than finding bifurcation points. points is easier for a computer.*

Here's a hint for the previous exercise: A superstable 2^n-cycle always contains the point $x = 1/2$ (Why?). Therefore to find μ_n, the parameter for which there is a superstable 2^n-cycle, we need only solve

$$\left(L_\mu^{2^n}\right)\left(\tfrac{1}{2}\right) = \tfrac{1}{2}.$$

for μ. And that's easy for a computer to do–you can use Newton's method.

The trouble arises that using Newton's method on this equation may not get the answer you want. Note μ_m is a superstable 2^n-cycle for all $n \geq m$; just not a prime one. So in trying to find μ_n, we may get drawn to a lower μ_m which we know already. To fix this, we have to give a good guess to start with. Define

$$\delta_n = \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}$$

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Exercise 3. *Estimate δ_2 by zooming in on your bifurcation diagram. You will have to estimate graphically (or algebraically) the values for μ_1, μ_2, and μ_3, for which there exist superstable two-, four-, and eight-cycles. But remember that you are looking for the intersection of the region of 2^n-fold periodicity with a certain horizontal line.*

Given this approximation, and assuming the δ_n converge, we can conclude that

$$\mu_n \approx \mu_{n-1} + \frac{\mu_{n-1} - \mu_{n-2}}{\delta_{n-1}}.$$

So the number on the right-hand side is a good initial value to feed the iterative map that finds the number on the left.

Exercise 4. *Find the values of μ_n and δ_n for n up to 10 or 11 (to get δ_{10} you need μ_{11}).*

At this point we are exposed to the seamy underbelly of *Mathematica*. Up to now we've been doing all of our calculations symbolically. For instance, calculating $L_\mu^n(0.5)$ involves generating a polynomial in x of degree 2^n, and then "plugging in" 0.5 to this polynomial. This takes a lot of space (memory) as n gets large. So if you try to stick to symbolic methods, *Mathematica* will run out of memory too quickly.

To combat this, we have to *compile* our functions. This will make them faster and will not take up so much space. The down side is that they can not longer be manipulated symbolically. The on-line documentation of `Compile` is very helpful. You will want to compile functions for $L_\mu^n(x)$ as a function of μ, x, and n (use a loop rather than a recursive algorithm), as well as $\frac{\partial L_\mu^n}{\partial \mu}$. For the latter, you can approximate by a difference quotient:

$$\frac{\partial L_\mu^n}{\partial \mu}(x) \approx \frac{L_{\mu+\varepsilon}^n(x) - L_{\mu-\varepsilon}^n}{2\varepsilon}.$$

ε can be a constant as long as it's very small, or you can have it vary with n.

Return to the orbit diagram. A second scaling factor is in "state space" represented by the x coordinate. What we want is the *separation* of the superstable 2^n-cycle.

Exercise 5. *Let $\{x_0, \dots, x_{2^n}\}$ be a superstable 2^n-cycle, with $x_0 = 1/2$. Which of the other points is closest to x_0 and why?*

Let d_n be this minimum width, $d_n = x_m - 1/2$, where m is the number such that x_m is closest to $1/2$. What we are looking for is the limiting ratio

$$\alpha = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{d_{n-1}}{d_n}.$$

Exercise 6. *Compute an approximation to α.*

Exercise 7. *Compute the numbers δ and α for the family $S_\lambda(x) = \lambda \sin \pi x$ for $0 \le \lambda \le 1$. Your code should be flexible enough so that you don't have to change much to do this!*

REFERENCES

[Fei80] Mitchell J. Feigenbaum, *Universal behavior in nonlinear systems*, Los Alamos Science **1** (1980), 4–27.

[Vin93] Michael Vinson, *Scaling and universality*, Unpublished course text, 1993.