MATH 118 : SPRING 1999
MIDTERM EXAMINATION SOLUTIONS

MATTHEW LEINGANG, GCA

Problem 1. (a) *Define what it means for $T\colon X \to X$ to be a contraction on a space X with metric d. State the important theorem we proved about such contraction mappings.*

(b) *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable map. Give a condition (with proof) on f' that guarantees that f is a contraction.*

Answer. (a) A *contraction* on X is a map $T\colon X \to X$ for which there exists $\alpha < 1$ such that for all $x, y \in X$,

$$d\big(T(x), T(y)\big) \leq \alpha d(x, y).$$

The important theorem is known as the *Contraction Mapping Theorem*, which says that if X is complete, then T has a unique fixed point $p \in X$, and moreover, for all $x \in X$,

$$\lim_{n\to\infty} T^n(x) = p.$$

(b) Suppose that there exists $\alpha < 1$ such that $|f'| \leq \alpha$. Let $x, y \in R$. Then

$$|f(x) - f(y)| = |f'(z)|\,|x - y|,$$

for some z between x and y, by the Mean Value Theorem. Then by our supposition,

$$|f(x) - f(y)| \leq \alpha\,|x - y|,$$

which means that f is a contraction.

□

Remark 1. It is *not* enough to suppose that $|f'| < 1$. Consider the function $f(x) = \frac{1}{2}\left(x + \sqrt{4 + x^2}\right)$. Then $f'(x) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{4+x^2}}\right) < 1$, but f has no fixed points, so in particular can't be a contraction. See Figure 1.

Problem 2. (a) *Let $F\colon \mathbb{R} \to \mathbb{R}$ be a differentiable map. Define what it means for p to be a hyperbolic fixed point for F.*

(b) *Let F_λ be a one-parameter family of maps. Describe a saddle-node bifurcation of this family at some parameter value λ_0.*

(c) *Prove that if F_{λ_0} has a hyperbolic fixed point then there is no saddle-node bifurcation at x_0.*

Answer. (a) A *hyperbolic fixed point* $p \in \mathbb{R}$ enjoys $F(p) = p$ and $|F'(p)| \neq 1$.

(b) A *saddle-node bifurcation* occurs at λ_0 if there exists $\varepsilon > 0$ and an interval $I \subset \mathbb{R}$ such that

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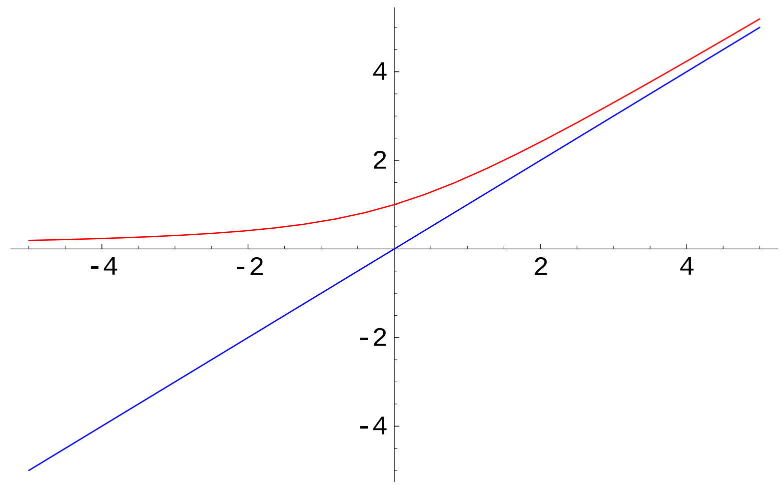

FIGURE 1. A function $f\colon \mathbb{R} \to \mathbb{R}$ with $|f'| < 1$ but f is not a contraction.

(1) for $\lambda_0 - \varepsilon < \lambda < \lambda_0$, there are no fixed points for F_λ in I;
(2) F_{λ_0} has a unique fixed point in I. It follows from Problem 2(c) that this fixed point must be nonhyperbolic.
(3) For $\lambda_0 < \lambda < \lambda + \varepsilon$, F_λ has two fixed points in I, one which is attracting and one which is repelling.

Cf. [Dev92, Section 6.2]. An example is the exponential family $E_\lambda(x) = e^x + \lambda$ at $\lambda = -1$. See Figure 2.

(c) Suppose that the fixed point p of F_{λ_0} is nonhyperbolic. Then define $P(\lambda, x) = F_\lambda(x) - x$. Note that $P(\lambda_0, p) = 0$, and

$$\frac{\partial P}{\partial x}(\lambda_0, p) = F'_{\lambda_0}(p) - 1 \neq 0.$$

So by the Implicit Function Theorem, there exists a local solution $x(\lambda)$ of fixed points for F_λ. This means that there can be no bifurcation at λ_0. Cf. [SG99, Section 2.2].

□

Problem 3. *Consider* $f(x) = -3\left|x - \frac{1}{2}\right| + \frac{3}{2}$.

(a) *What is the set* Y *of points* $y \in \mathbb{R}$ *such that* $f^n(y) \not\to -\infty$ *as* $n \to \infty$*?*
(b) *Show that* $f\colon Y \to Y$ *is chaotic by first exhibiting a conjugacy between this map and a map you know is chaotic, and then showing this conjugacy is continuous.*

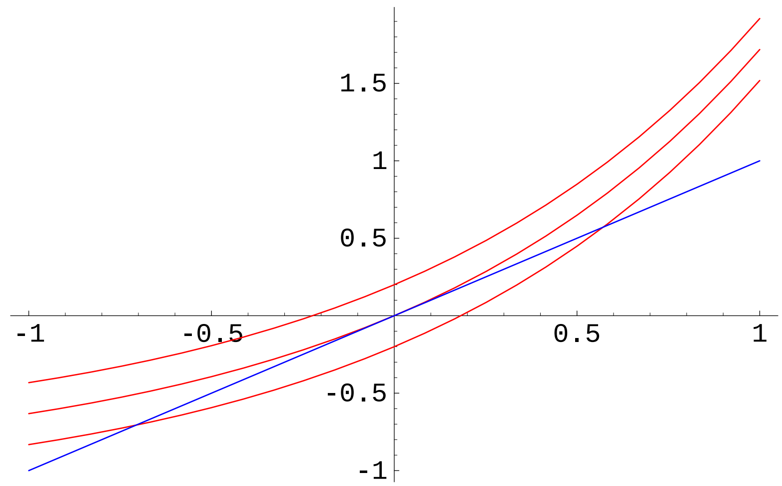

FIGURE 2. The saddle-node bifurcation of the exponential family $E_\lambda(x) = e^x + \lambda$ at $\lambda = -1$.

(c) *Justify this conclusion further by defining what it means in general for a map g to be chaotic on a metric space X, and then showing directly that the map f is conjugate to really does satisfy* one *of the required conditions.*

Answer. We can rewrite f as

$$f(x) = \begin{cases} 3x & \text{if } x \le \frac{1}{2}; \\ 3(1-x) & \text{if } x \ge \frac{1}{2}. \end{cases}$$

(a) Let $Y_n = f^{-n}[0,1]$. Then $Y = \bigcap_{n\ge 1} Y_n$. We claim by induction that Y_n is a finite collection of closed intervals. Morever, if $[a,b]$ is a component, i.e., a maximal closed interval in Y_n with $f^n(a) = 0$ and $f^n(b) = 1$, then

$$[a,b] \cap Y_{n+1} = f^{-1}[a,b] = \left[a + \frac{b-a}{3}, b - \frac{b-a}{3}\right].$$

Then Y is obviously the Cantor middle-thirds set.

(b) Let Σ be the sequence space on two symbols. Let $I_0 = [0, \frac{1}{3}]$, $I_1 = [\frac{2}{3}, 1]$ in $\mathbb{R}$. Define $\varphi\colon Y \to \Sigma$ by the relation

$$f^n(y) \in I_{\varphi(y)_n}.$$

This makes sense because $f^n(Y) \subset I_0 \cup I_1$ for all n. Then also since

$$f^n\big(f(y)\big) = f^{n+1}(y) \in I_{\varphi(y)_{n+1}} = I_{\sigma(\varphi(y))_n},$$

and yet by definition $f^n\big(f(y)\big) \in I_{\varphi(f(y))_n}$, we see that $\sigma \circ \varphi = \varphi \circ f$. So φ is a conjugacy between the dynamical systems (Y, f) and (Σ, σ).

We claim that φ is continuous. To see this, recall the proximity theorem. Let $\varepsilon > 0$ be given, and choose N such that $2^{-N} < \varepsilon$. Then let $\delta = 3^{-N-2}$. Supose that $|x - y| < \delta$. Then since f is Lipschitz with Lipschitz constant 3, we have

$$|f^n(x) - f^n(y)| \leq 3^n \cdot 3^{-N-2},$$

which means that $f^n(x)$ and $f^n(y)$ are in the same subinterval of Y_n for $n = 0, \ldots, N$. This means that the first N digits of x and y agree. Thus $d\big(\varphi(x), \varphi(y)\big) \leq 2^{-N} < \varepsilon$. So φ is continuous.

(c) $g\colon X \to X$ is chaotic if
  (i) $\operatorname{Per} g$ is dense in X;
  (ii) g is topologically transitive.

The first of these is easy to verify for the shift map σ. Indeed, let $\mathbf{s} = (s_0 s_1 s_2 \ldots) \in \Sigma$ and $\varepsilon > 0$ be given. Again, choose $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon$. Then let

$$\mathbf{s}' = (\overline{s_0 s_1 \ldots s_N}).$$

Then $\mathbf{s}'$ is periodic of period $N + 1$ and $d(\mathbf{s}, \mathbf{s}') \leq 2^{-N} < \varepsilon$.

□

Problem 4. *Consider Newton's method for solving* $\sin x = 0$. *Let k be an integer.*

(a) *Does $W^s(k\pi)$ contain a neighborhood of $k\pi$? Prove your answer.*
(b) *Is $W^s(k\pi)$ bounded? Justify with a sketch.*

Answer. Note that $N_{\sin}(x) = x - \frac{\sin x}{\cos x} = x - \tan x$. If k is any integer, $\sin k\pi = 0$.

(a) Since $N'_{\sin}(k\pi) = 1 - \sec^2 k\pi = 0$, each $k\pi$ is a superattracting fixed point for $N_{\sin}$, hence attracts a neighborhood of itself.
(b) The answer is "no." A sketch illustrating the algorithm shows that there are points infinitely far from $k\pi$ that tend to it under iterates of $N_{\sin}$. First, note that there exists a neighborhood $U \ni k\pi$ consisting of points which tend to $k\pi$, so all we have to do is get into U. Consider the graph of sin. If we take an $x \in U$, there are infinitely many numbers μ near 0 such that the line through $(x, 0)$ with slope μ intersects the graph tangentially. These points are as far out as we need. See Figure 3.

□

Problem 5. *Consider the function* $f[1,5] \to [1,5]$ *that satisfies*

$$f(1) = 3;$$
$$f(2) = 5;$$
$$f(3) = 4;$$
$$f(4) = 2;$$
$$f(5) = 1,$$

and that is piecewise linear.

(a) *Show that f has no points of period three in the intervals* $[1,2]$, $[2,3]$, *and* $[4,5]$.
(b) *What about in the interval* $[3,4]$*?*

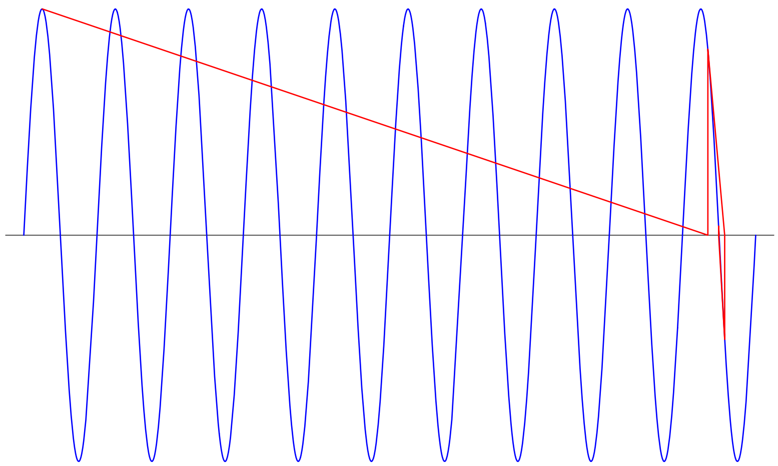

FIGURE 3. There are points arbitrarily far away from any root of sin that iterate to it.

(c) *State Šarkovskii's theorem and explain what this example illustrates about it.*

Answer. (a) See 4 for the graph of f. Note that $\{1,2,3,4,5\}$ is a cycle of *prime* period five. We just track the intervals under f:

$$[1,2] \to [3,5] \to [1,4] \to [2,5].$$

Thus $f^3[1,2] \cap [1,2] = \{2\}$, which belongs to a five-cycle. So there cannot be a point of period three in $[1,2]$. The same goes for $[2,3]$, since $f^3[2,3] = [3,5]$, and for $[4,5]$, since $f^3[4,5] = [1,4]$.

(b) Here we see that

$$[3,4] \to [2,4] \to [2,5] \to [1,5] \supset [3,4]$$

so f has a fixed point and a point of period three in [3,4]. But we claim that there are actually the same point. That is, the purported three-cycle is actually of prime period one. To prove this, note that f^3 is a composition of three decreasing maps when restricted to in intervals in questions. Hence f^3 is decreasing, and can only have a single fixed point.

(c) In the Šarkovskii ordering, $3 \triangleright 5$, that is, we must see period five whenever we see period three. We have just shown that this inequality is sharp. Namely, we have shown $5 \not\triangleright 3$.

□

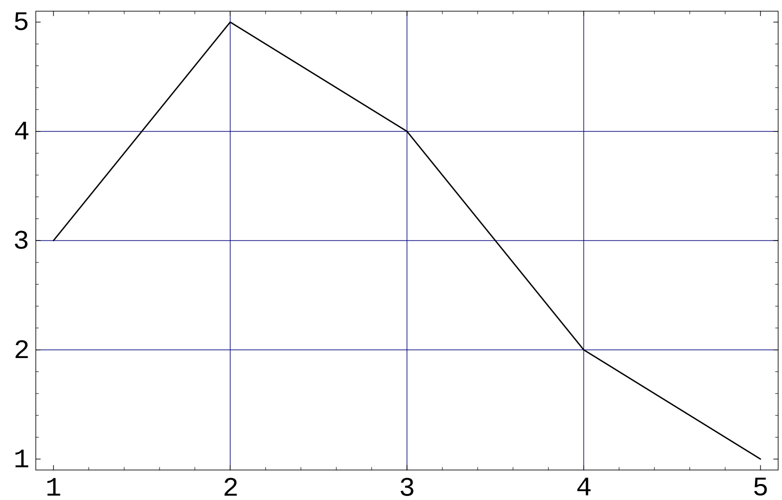

Figure 4. The graph of f from Problem 5.

References

[Dev92] Robert L. Devaney, *A First Course in Chaotic Dynamical Systems: Theory and Experiment*, Addison-Wesley, 1992.

[SG99] Shlomo Z. Sternberg and Daniel Goroff, *Math 118 Notes*, Harvard University Department of Mathematics, February 1999, Preliminary Version.