MATH 118
PROBLEM SET 7

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Exercise 1. For each of the following functions $v\colon \mathbb{R}^2 \to \mathbb{R}^2$, find the fixed points of

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = v\begin{pmatrix} x \\ y \end{pmatrix}$$

determine their stability, and sketch the phase portrait.

(a)

$$v\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^3 - y \\ x + y \end{pmatrix}$$

(b)

$$v\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin y \\ x + y \end{pmatrix}$$

(c)

$$v\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3y - e^x \\ 2x - y \end{pmatrix}$$

(d)

$$v\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x + y \end{pmatrix}$$

Exercise 2. Use a Lyapunov function to show that $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ is a stable fixed point for

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y^2 - x - 1 \\ -xy - y \end{pmatrix}.$$

Exercise 3. Apply the Picard iteration to solve $\frac{dx}{dt} = 1 + x^2$ with $x(0) = 0$. Show that after three Picard iterations, the reulst agrees with the true solution for terms of degree five or less in t. Can the solution to this equation be continued for all time?

Exercise 4. Show that $\frac{dx}{dt} = 3x^{2/3}$ and $x(0) = 0$ has infinitely many solutions given by

$$x(t) = \begin{cases} 0 & \text{if } t < c; \\ (t-c)^3 & \text{if } t \geq c. \end{cases}$$

Does $x^{2/3}$ satisfy a Lipschitz condition?

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Exercise 5. Exponentiate and sketchphase portraits for $\frac{d\vec{x}}{dt} = A\vec{x}$. Which are hyperbolic?

(a)

$$A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$$

(d)

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(e) Make up an interesting one of your own.