5/4/05 FINAL QUIZZ **Math118, O.Knill**

1) Which of the following dynamical systems have a discrete time? We replace "map" or "differential equation" with "system".

☐ a) Henon system
☐ b) Geodesic system
☐ c) Cellular automata system

2) What is a semigroup?

☐ a) A set G with an operation $*$.
☐ b) A set G with an operation $*$ such that $(x \star y) \star z = x \star (y \star z)$.
☐ c) A set G with an operation $*$ such that $(x \star y) \star z = x \star (y \star z)$ with a neutral element e satisfying $x \star e = x$.

3) Which of the following dynamical systems allow a numerical computation of the square root of 7:

☐ a) $T(x) = \sqrt{x} - 7$.
☐ b) $T(x, y) = ((x + y)/2, 2xy/(x + y))$.
☐ c) $T(x) = x^2 + 7$.

4) There is a sentence attributed to Steven Smale which appears also in the movie "Jurassic Park". The statement is "The wing of a butterfly in X can produce a tornado in Y a few weeks later":

☐ a) X=New York, Y = Los Angeles
☐ b) X=Rio, Y=Texas
☐ c) X=Chicago, Y = New Orleans

5) We have seen that for a computer, iterations of the map $T(x) = 4x(1 - x)$ or the map $S(x) = 4x - 4x^2$ give different results for $T^n(x)$ and $S^n(x)$ if n is large and this happened for identical initial condition x_0. For which n, did we see different results?

☐ a) $n = 1$
☐ b) $n = 10$
☐ c) $n = 100$

6) Which of the following properties do hold at a **saddle node bifurcation**?

☐ a) $f'(x) = 1$
☐ b) $(f^2)'(x) = -1$
☐ c) The bifurcation is also called **blue-sky** bifurcation

7) The tent map $g(x) = 1 - 2|x - 1/2|$ is conjugated to the logistic map $f_c(x) = cx(1 - x)$ for the parameter

☐ a) $c = 0$
☐ b) $c = 2$
☐ c) $c = 4$

8) Which of the following formulas do give the Lyapunov exponent of an orbit $x_0, x_1, x_2, ...$ of a map $f : [0, 1] \to [0, 1]$?

☐ a) $\lambda(f, x_0) = \lim_{n\to\infty} \log |(f^n)'(x_0)|$
☐ b) $\lambda(f, x_0) = \lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x_0)|$
☐ c) $\lambda(f, x_0) = \lim_{n\to\infty} \frac{1}{n} |(f^n)'(x_0)|$

9) Which properties can change at a bifurcation point?

☐ a) the number of periodic points
☐ b) the stability of periodic points
☐ c) the sign of $\log |f'(x)|$

10) Which of the following things are true:

☐ a) 1/2 is an eventually periodic orbit of the Ulam map $f(x) = 4x(1 - x)$
☐ b) 1/2 is a periodic orbit of the Ulam map
☐ c) The Lyapunov exponent $\lambda(f, 1/2)$ is equal to $\log(2)$ if f is the Ulam map

11) One of the following maps is an area preserving Henon map. Which one?

☐ a) $T(x, y) = (3(1 - x^2) - y, 2x)$
☐ b) $T(x, y) = (2x - y + \sin(x), x)$
☐ c) $T(x, y) = (3(1 - x^2) - y, x)$

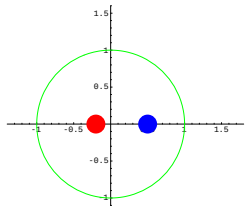
12) For a certain parameters c, the Standard map $T(x, y) = (2x + c \sin(x) - y, x) \mod 1$ is integrable. Which integral verifies this fact?

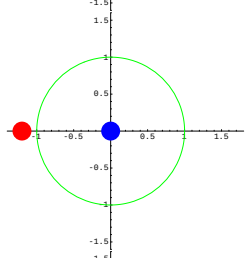
☐ a) $F(x, y) = x$
☐ b) $F(x, y) = x - y$
☐ c) $F(x, y) = 1$

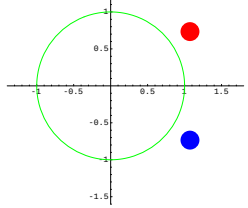
13) A map in the plane is called an **involution** if $T^2 = Id$, that is if every point is periodic with period 2. Which of the following statements are true?

☐ a) In general, an involution is integrable
☐ b) The map $T(x, y) = (-x, y + c \sin(x))$ is an involution
☐ c) All linear involutions are area-preserving

14) Which of the following 3 pictures shows the eigenvalues of a Jacobean at a repelling fixed point of a map T in the plane?

☐ 

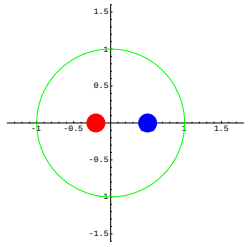
☐ 

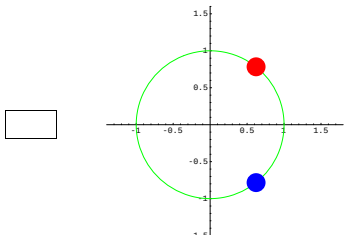
☐ 

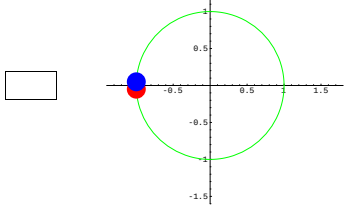
15) If the stable and unstable manifolds of a hyperbolic fixed point (x_0, y_0) of intersect transversely, then this intersection point is called

☐ a) a horse shoe
☐ b) an integral
☐ c) a homoclinic point

16) Which of the following 3 pictures show the eigenvalues of an asymptotically stable equilibrium point (x_0, y_0) (sink) of the differential equation $\dot{x} = F(x)$.

☐ 

☐ 

☐ 

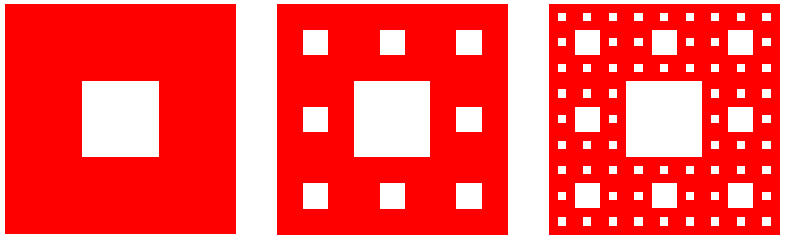
17) Which of the following formulations is the correct version of the Poincaré Bendixson theorem?

☐ a) An orbit in the plane which stays in a bounded region is either asymptotic to an equilibrium point or to a limit cycle
☐ b) An orbit in the plane which is not asymptotic to a limit cycle is attracted to an equilibrium point
☐ c) Every orbit of a differential equation in the plane is either asymptotic to a limit cycle or to an equilibrium point

18) A differential equation of the form $\frac{d}{dt}x(t) = H_y(x, y)$, $\frac{d}{dt}y(t) = -H_x(x, y)$, where $H(x, y)$ is a function of two variables.

☐ a) produces an area-preserving flow
☐ b) is integrable
☐ c) has an attractive limit cycle

19) We define an object in the plane by cutting away 1 square of length 1/3 from a square of length 1, then repeating this construction with remaining squares of length 1/3 etc. The first three steps are shown here:



What is the dimension of this object?

☐ a) $\log(8)/\log(3)$
☐ b) $\log(14)/\log(4)$
☐ c) $\log(20)/\log(3)$

20) The billiard in an ellipse is known to be integrable. What is the integral F?

☐ a) The sum of the shortest distances of the trajectory line to the focal points
☐ b) The product of the shortest distances of the trajectory line to the focal points
☐ c) The impact angle θ

21) Which of the following formulations is the full content of the Hedlund-Lyndon-Curtis theorem? (only one answer is correct).

☐ a) Any map T on $X = \{0,1\}^Z$ which is continuous and commutes with the shift is of the form $T(x)_n = \phi(x_{n-1}, x_n, x_{n+1})$
☐ b) A cellular automaton is a continuous map on $X = \{0,1\}^Z$
☐ c) A shift commuting, continuous map on $X = \{0,1\}^Z$ is a cellular automaton

22) We have $T(x)_n = x_{n+1} + x_{n-1} + x_n \mod(2)$. What is the image of the sequence $x = (...,1,1,1,1,1,1,1,....)$?

☐ a) (...,1,1,1,1,1,1,1,1,1,1,1,1,...)
☐ b) (...,0,0,0,0,0,0,0,0,0,0,0,0,...)
☐ c) (...,1,0,1,0,1,0,1,0,1,0,1....)

23) What is a "glider" in the game of life (X, T)?

☐ a) A configuration x which satisfies $T^n(x) = \sigma^m(x)$ for $n, m > 0$
☐ b) A configuration with finitely many living cells which satisfies $T^n(x) = \sigma^m(x)$ for $n, m > 0$
☐ c) A configuration which satisfies $T^n(x) = x$ for $n > 0$

24) Who is believed to have first come up with the notion of cellular automata?

☐ a) Hedlund at Harvard
☐ b) Wolfram at Caltech
☐ b) Ulam and von Neuman at Los Alamos

25) With $f_c(z) = z^2 + c$, which of the following statements are true?

☐ a) The Mandelbrot set is the set of complex numbers such that $f_c^n(c) \to \infty$
☐ b) The Mandelbrot set is the set of complex numbers such that $|f_c^n(c)| \leq 2$ for all n.
☐ c) The Mandelbrot set is the set of complex numbers such that $f_c^n(c)$ stays bounded.

26) Which of the following statements are true?

☐ a) Every quadratic polynomial is conjugated to the polynomial $f_c(z) = z^2 + c$
☐ b) Every cubic polynomial is conjugated to a polynomial $g_{a,b}(z) = z^3 - 3a^2z + b$
☐ c) Every cubic polynomial is conjugated to a polynomial $g_c(z) = z^3 + c$

27) Which of the following statements are true?

☐ a) The union of the filled in Julia set and the Fatou set is the entire complex plane
☐ b) The Fatou set is the complement of the Mandelbrot set.
☐ c) The Julia set always connected.

28) Which of the following dynamical systems is called the **Newton iteration** to find the root $f(z) = 0$:

☐ a) $T(z) = 1 - f(z)/f'(z)$
☐ b) $T(z) = z - f'(z)/f(z)$
☐ c) $T(z) = 1 - f'(z)/f(z)$

29) In order to find a fixed point of a map S, we can try to apply the Newton method to one of the following:

☐ a) $T(z) = S(z) - z$
☐ b) $T(z) = S'(z) - z$
☐ c) $T(z) = z - S(z)/S'(z)$

30) Which of the following properties apply to the Baker transformation T on the square $[0,1) \times [0,1)$.

☐ a) The map is continuous
☐ c) The map is area-preserving
☐ c) The map has many periodic points

31) What can you say about the subshift X of finite type over the alphabet $\{a, b, c\}$ defined by the forbidden words $\{aa, bb, cc, ac, ba, cb\}$?

☐ a) X does not contain any point
☐ b) X contains only finitely many points
☐ c) X contains infinitely many points

32) Which of the following subshifts is the shift over the alphabet $\{a, b\}$ for which all words $bab, baaab, baaaaab, baaaaaaab, baaaaaaaaab, ...$ etc. are forbidden?

- [ ] a) The Fibonacci shift
- [ ] b) The even shift
- [ ] c) The full shift

33) What is special about the **golden ratio** $\theta = (\sqrt{5} - 1)/2$? Check everything which applies:

- [ ] a) It has the smallest possible continued fraction expansion $\theta = [a_0; a_1, a_2, a_3, ...]$
- [ ] b) The partial fractions p_n/q_n have the property that q_n and p_n grow like Fibonnacci numbers
- [ ] c) θ is a solution of $x = 2/(2 + x)$

34) Which of the following statements is called Dirichlets theorem for an irrational number α.

- [ ] a) For every n, there exists $q \leq n$ such that
$$|\alpha - p/q| \leq 1/q^2 .$$
- [ ] b) For every n, there exists $q \leq n$ such that
$$|\alpha - p/q| \leq 1/q .$$
- [ ] c) For every q, there exists p such that
$$|\alpha - p/q| \leq 1/q^2 .$$

35) Which of the following three formulas give the correct recursion for the partial fractions p_n/q_n:

- [ ] a) $p_{n+1} = a_n p_n + p_{n-1}$, $q_{n+1} = a_n q_n + q_{n-1}$
- [ ] b) $p_{n+1} = p_n + a_n p_{n-1}$, $q_{n+1} = q_n + a_n q_{n-1}$
- [ ] c) $p_{n+1} = a_n p_n + a_{n-1} q_{n-1}$, $q_{n+1} = a_n p_n + a_{n-1} q_{n-1}$

36) Which dynamical system is involved when making a decimal expansion of a real number.

- [ ] a) $T(x) = 10x$
- [ ] b) $T(x) = 10x \bmod 1$
- [ ] c) $T(x) = 10/x \bmod 1$

37) The quadratic map $T(x) = x^2 + c$ is also useful in number theory. Where?

- [ ] a) to compute the eclipse times in calendars
- [ ] b) to factor large integers
- [ ] c) to understand why our tonal system has 12 scales between an octave and 19 scales for a perfect fifth

38) When finding lattice points close to graphs of quadratic polynomials, we were led to a dynamical system on the two dimensional torus. This system is

- [ ] a) $(x, y) \to (2x + y, x + y) \bmod 1$
- [ ] b) $(x, y) \to (x + 2a, x + y) \bmod 1$
- [ ] c) $(x, y) \to (x + y, x - y) \bmod 1$

39) Check for whatever continued fractions are useful:

- [ ] a) To compute eclipse cycles
- [ ] b) To find lattice points close to lines in the plane
- [ ] c) To justify why we use a 12 scale system in music

40) Which of the following statements are true?

- [ ] a) there are always equilibrium solutions to the Newtonian n-body problem
- [ ] b) the Sitnikov problem is a restricted three body problem
- [ ] c) the Lagrange solution is a periodic solution of the three body problem which has been observed in our solar system

41) If for all t, we have a skew-symmetric matrix $B(t)$ and $S(t)$ satisfies the matrix differential equation $\dot{S} = BS$ with the initial condition $S(0) = I$, then S is

- [ ] a) orthogonal
- [ ] b) skew symmetric
- [ ] c) symmetric

42) Which of the following statements is called the third Kepler law:

- [ ] a) The radius vector covers equal area in equal time
- [ ] b) Each of the bodies moves on an ellipse
- [ ] c) T^2/a^3 is constant

43) Which of the following forces occur in a rotating coordinate system and depend on the angular speed of the rotation?

- [ ] a) Centrifugal force
- [ ] b) Coriolis force
- [ ] c) Gravitational force

44) On which of the following surfaces is every geodesic periodic?

- [ ] a) The sphere
- [ ] b) The flat torus
- [ ] c) The torus embedded in space (the doughnut)