Calculating square roots with dynamical systems

Details to a result mentioned in the lecture of February 4'th 2005. The example can also found in the book (introduction).

Math118r O.Knill

“Theorem”:

Define $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x+y}{2} \\ \frac{2xy}{x+y} \end{bmatrix}$

With the initial condition $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ 1 \end{bmatrix}$

The orbit converges to $\begin{bmatrix} \sqrt{a} \\ \sqrt{a} \end{bmatrix}$

exponentially fast

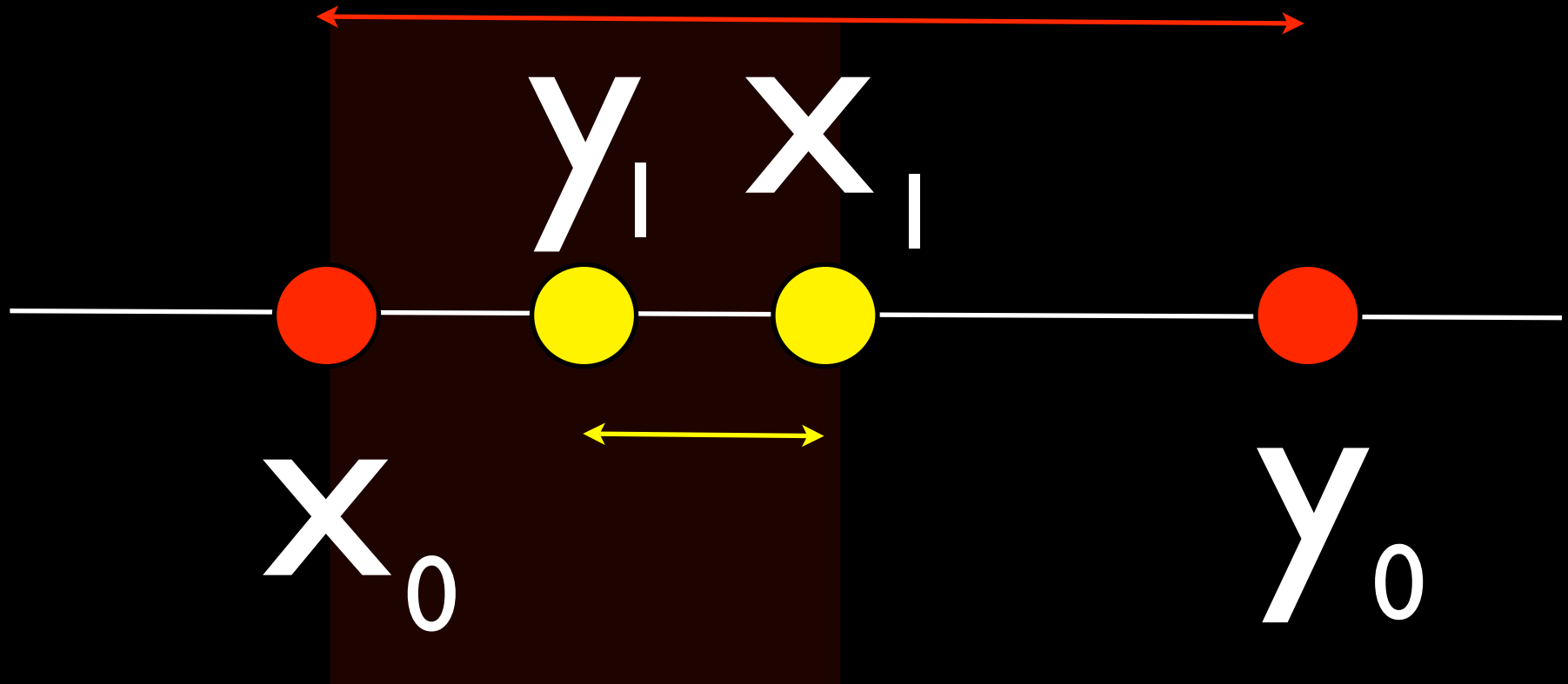
An integral:

The function $F(x,y) = x\ y$ is an integral.

It does not change after applying the map.

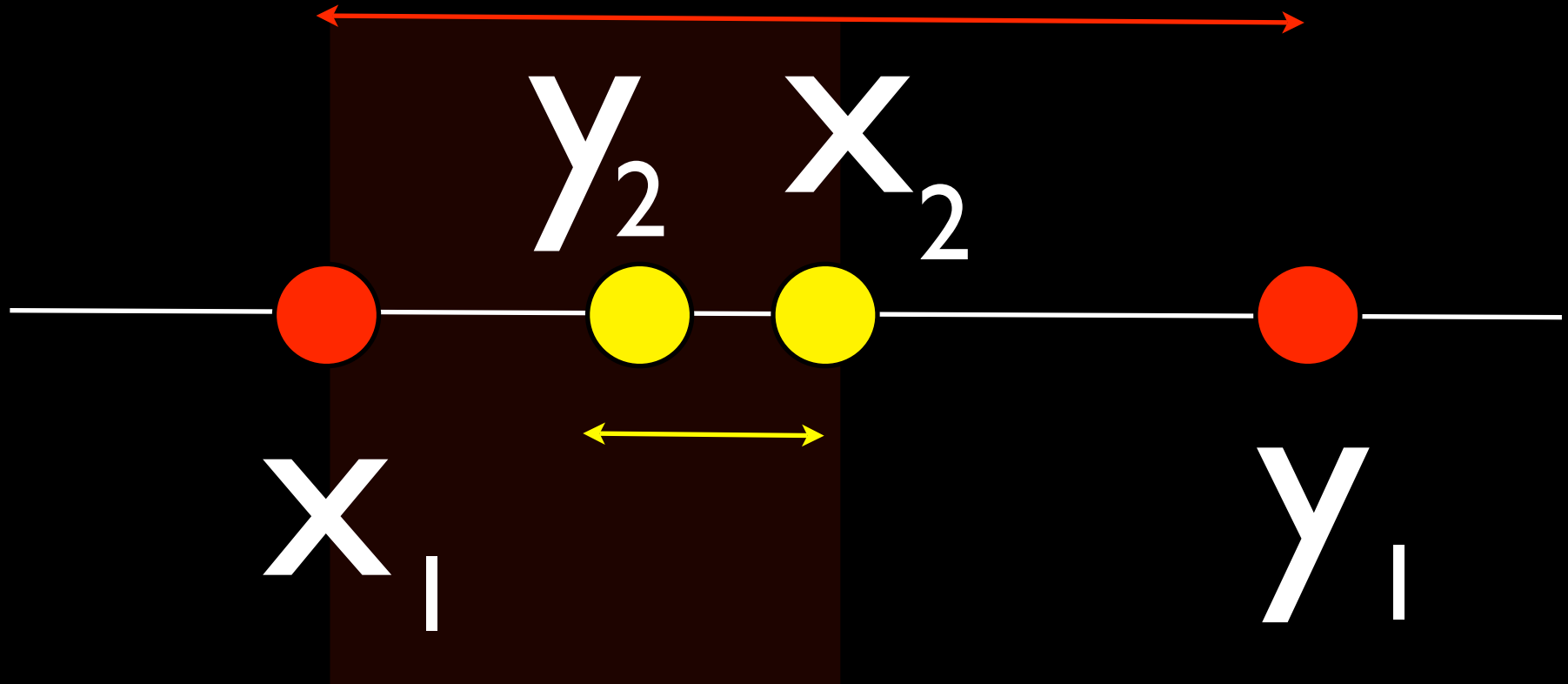
Key observation:

$$|y_1 - x_1| < |y_0 - x_0|/2$$



It follows that

$$|y_2 - x_2| < |y_0 - x_0|/4$$



etc. and by induction

it follows that

$$|y_n - x_n| < \frac{1}{2^n} |y_0 - x_0|$$

Conclusion:

Because $x_n y_n = a$ is constant and $|x_n - y_n|$ goes to zero exponentially fast, we must have convergence to the square root of a.