VLASOV DYNAMICS (*) — Math118, O. Knill

ABSTRACT. Vlasov dynamics generalizes n-body particle dynamics. But there is a geometric twist.

VLASOV DYNAMICS. Let M be a manifold of dimension p with measure m and cotangent bundle T^*M and let N be a manifold of dimension q with cotangent bundle T^*N. Take $N = \mathbf{R}^q, T^*M = \mathbf{R}^{2q}$. A one-parameter family of maps $X^t = (f^t, g^t) : T^*M \to T^*N$ is defined by the differential equation

$$\boxed{\dot{f} = g, \dot{g} = -\int_{T^*N} \nabla V(f(\omega) - f(\eta))\, dm(\eta)}$$

where V is a potential. This is a Hamiltonian system.

EXAMPLES. 1) If T^*M is zero dimensional with n points $\{\omega_1, \dots, \omega_n\}$, then X^t describes the evolution of n particles $(f_i, g_i) = X(\omega_i)$. Vlasov dynamics is therefore a generalisation of n-body dynamics.
2) If $N = M$, then X^t are volume-preserving deformations of T^*M.

FACT. If (f, g) move according to $\dot{f} = g, \dot{g} = -\int \nabla V(f(\omega) - f(\eta))\, dm(\eta)$, then the density $P = (f, g)^* m$ satisfies the Vlasov equation $\boxed{\dot{P}(x, y, t) + y\nabla_x P^(x, y) - E(x)\nabla_y P^t(x, y) = 0}$ with $E(x) = \int_{T^*N} \nabla_x V(x - x') \cdot P^t(x', y'))\, dy' dx'$.

PROOF. We have $\int \nabla V(f(\omega) - f(\eta))\, dm(\eta) = E(f(\omega))$. Given a smooth function h on T^*N of compact support. We have $\int_{T^*N} h(x, y) \frac{d}{dt} P^t(x, y)\, dxdy = \frac{d}{dt} \int_{T^*N} h(x, y)P^t(x, y)\, dxdy = \frac{d}{dt} \int_M h(f(\omega, t), g(\omega, t))\, dm(\omega) = \int_M \nabla_x h(f(\omega, t), g(\omega, t))g(\omega, t)\, dm(\omega) - \int_M \nabla_y h(f(\omega, t), g(\omega, t)) \int_M \nabla V(f(\omega) - f(\eta))\, dm(\eta)\, dm(\omega) = \int_{T^*N} \nabla_x h(x, y)yP^t(x, y)\, dxdy - \int_{T^*N} P^t(x, y)\nabla_y h(x, y) \int_{T^*N} \nabla V(x - x')P^t(x', y')\, dx'dy'dxdy = -\int_{T^*N} h(x, y)\nabla_x P^t(x, y)y\, dxdy + \int_{T^*N} h(x, y)E(x) \cdot \nabla_y P^t(x, y)\, dxdy$.

EXAMPLES.
1) $V(x) = 0$. Particles move freely. The transport equation is $\dot{P}(x, y, t) + y \cdot \nabla_x P^t(x, y) = 0$ which has solutions $P^t(x, y) = P^0(x + ty)$.

2) For a quadratic potential $V(x) = x^2$, the Hamilton equations are $\ddot{f}(\omega) = -(f(\omega) - \int_M f(\eta)\, dm(\eta))$. In center-of-mass-coordinates $f \mapsto f - \int f dm$, the system is a decoupled system of a continuum of oscillators $\dot{f} = g, \dot{g} = -f$ with solutions $f(t) = f(0)\cos(t) + g(0)\sin(t)$, $g(t) = -f(0)\sin(t) + g(0)\cos(t)$. The evolution for the density P is the partial differential equation $\dot{P}(x, y, t) + y \cdot \nabla_x P^t(x, y) - x \cdot \nabla_y P^t(x, y) = 0$ which has the explicit solution $P^t(x, y) = P^0(\cos(t)x + \sin(t)y, -\sin(t)x + \cos(t)y)$.

3) (advanced) On a Riemannian manifold with Laplace-Beltrami operator Δ, the Poisson equation $\Delta\phi = \rho$ is solved by $\phi = V \star \rho$.

- $N = \mathbf{R}$: $V(x) = |x|$.
- $N = \mathbf{T}$: $V(x) = |x(\pi - x)|$
- $N = \mathbf{S}^2$. $V(x) = \log(1 - x \cdot x)$.
- $N = \mathrm{R}^2$ $V(x) = \log(x)$.
- $N = \mathrm{R}^3$ $V(x) = 1/|x|$.
- $N = \mathrm{R}^4$ $V(x) = 1/|x|^2$.

4) $V(x) = x^{-2}$ Calogero-Moser potential.

BATT-NEUNZERT-BROWN-HEPP-DOBRUSHIN EXISTENCE THEOREM.

If $\nabla_x V$ is bounded and globally Lipshitz continuous, then $\ddot{f}(\omega) = -\int_M \nabla V(f(\omega) - f(\omega'))\, dm(\omega')$ has a unique global solution. Consequently the Vlasov equation has a unique and global solution in the space of measures. If V and P_0 are smooth, then P^t is piecewise smooth.

PROOF.

Take $M = T^*N$ and let $m = P_0$ be the initial measure. The Hamiltonian differential equation for $X = (f, g)$ on the complete metric space of all continuous maps from M to T^*N, which is a Banach manifold over the Banach manifold $C^r(M, TT^*N)$. The distance is $d(h, h') = \sup_{\omega \in M} d(h(\omega), h'(\omega))$ and if h, h' are in the same chart, we write also $||h - h'||_\infty = d(h, h')$.

With $X^0 = Id$, the initial data $(f_0, g_0)(x) = x$, we have $P_0 = (f_0, g_0)^* m$. The differential equation $\dot{f} = g$ and $\dot{g} = G(f) = -\int_M \nabla_x V(f(\omega) - f(\eta))\, dm(\eta)$ in $C(M, T^*N)$ has a unique solution: because of Lipshitz continuity

$$||G(f) - G(f')||_\infty \leq 2||D(\nabla_x V)||_\infty \cdot ||f - f'||_\infty$$

which holds in a neighborhood chart of f. The standard Piccard existence theorem for differential equations in Banach manifolds assures local existence.

The global Lipshitz assumption and a Gronwall estimate assures that $||X(\omega)||$ can not grow faster than exponentially leading to global existence.

The result could also be derived from the existence theorem applied to finite measure where the evolution is a n-body evolution. Uniqueness and global existence of solutions on a dense set of point measures implies uniqueness in general if the dynamics depends continuously on the measure m.

LINEARIZED MOTION.

The evolution of DX^t at a point $\omega \in M$ is $D\ddot{f}(\omega) = -\int_M \nabla^2 V(f(\omega) - f(\eta))\, dm(\eta) Df(\omega) =: B(f^t)Df(\omega)$ and

$$\frac{d}{dt} DX = \frac{d}{dt} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \int_M -\nabla^2 V(f(\omega) - f(\eta))\, dm(\eta) & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = A(f^t)DX \, .$$

CRITICAL POINTS of f.
The rank of the matrix $DX(\omega)$ stays constant. $Df^t(\omega)$ is a linear combination of $Df^0(\omega)$ and $D\dot{f}^0(\omega)$. Critical points of f^t can only appear for ω, where $Df^0(\omega), D\dot{f}^0(\omega)$ are linearly dependent. More generally $Y_k(t) = \{\omega \in T^*M \mid DX^t(\omega) \text{ has rank } 2q - k = dim(T^*N) - k\}$ is time independent. The set Y_q contains $\{\omega \mid D(f)(\omega) = \lambda D(g)(\omega), \lambda \in \mathbf{R} \cup \{\infty\}\}$.

LYAPUNOV EXPONENT.

$\lambda(\omega) = \limsup_{t \to \infty} t^{-1} \log(||D(X^t(\omega)||) \in [0, \infty]$. $\lambda(\omega)$ is the maximal Lyapunov exponent of the $SL(2q, \mathbf{R})$-cocycle $A^t = A(f^t)$ along an orbit (f^t, g^t). The Lyapunov exponent could be infinite.

HESSIAN. Differentiation of $D\ddot{f} = B(f^t)f^t$ at a critical point ω^t gives $D^2\ddot{f}^t(\omega^t) = B(f^t)D^2 f^t(\omega^t)$ The eigenvalues λ_j of the Hessian $D^2 f$ satisfy $\ddot{\lambda}_j = B(f^t)\lambda_j$.

EQUILIBRIUM MEASURES. Equilibrium measures are stationary solutions of the Vlasov equation. One can get them with a Maxwellian ansatz $P(x, y) = C\exp(-\beta(\frac{y^2}{2} + \int V(x - x')Q(x')\, dx)) = S(y)Q(x)$, C is chosen such that $\int_{\mathbf{R}^d} S(y)\, dy = 1$. They are called **Bernstein-Green-Kruskal** (BGK) modes.
If $Q : N \mapsto \mathbf{R}$ satisfies the integral equation $Q(x) = \exp(-\int_{\mathbf{R}^d} \beta V(x - x')Q(x')\, dx' = \exp(-\beta V \star Q(x))$ then the Maxwellian distribution $P(x, y) = S(y)Q(x)$ is an equilibrium solution of the Vlasov equation to the potential V because $y\nabla_x P = yS(y)Q_x(x) = yS(y)(-\beta Q(x) \int_{\mathbf{R}^d} \nabla_x V(x - x')Q(x')\, dx')$ and

$$\int_{T^*N} \nabla_x V(x - x')\nabla_y P(x, y)P(x', y')\, dx'dy' = Q(x)(-\beta S(y)y) \int \nabla_x V(x - x')Q(x')\, dx'$$

gives $y\nabla_x P(x, y) = \int_{T^*N} \nabla_x V(x - x')\nabla_y P(x, y)P(x', y')\, dx'\, dy'$.