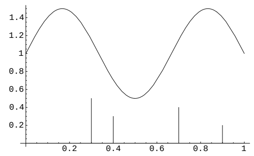
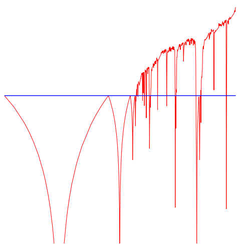
ABSTRACT.

INVARIANT MEASURES. A bounded linear map ϕ on the set $C(X)$ of all continuous functions on $X = [0,1]$is called a **measure**. It is called **positive** if $\phi(f) \geq 0$ if $f \geq 0$. One writes also $\mu(f) = \int f \; d\mu$. By approximation of $1_{a,b}$ by continuous functions f_n (for example with Fourier series), one can define $\mu([a,b]) = \int_a^b \; d\mu = \int_0^1 f_n(x) \; d\mu(x)$ and think of μ as a function defined on a set of subsets of the real lines. If μ is a positive measure and $\mu(1) = 1$ if $1(x) = 1$ is the function which is constant 1, then μ is called a **probability measure**. A measure is called g**-invariant** if $\int g(f(x)) \; d\mu(x) = \int f(x) \; d\mu(x)$. The measure $\mu(g) = \frac{1}{n} \sum_{k=1}^n g(x_k)$ where $x_0, x_1, ..., x_n$ is an n-cycle is an invariant measure. A T-invariant measure is called **ergodic**, if all invariant sets have measure zero or one.



EXAMPLES.
1) The map $T(x) = x + \alpha$ mod 1 has the Lebesgue measure dx invariant. The Lebesgue measure is defined by the usual integration $\phi(f) = \int_0^1 f(x) \; dx$ is a probability measure. If α is irrational, then it is ergodic.
2) If $T(x) = 4x(1-x)$ on $I = [0,1]$, then $dx/(\pi\sqrt{x(1-x)})$ is an invariant measure. It is known to be ergodic.
3) If $T(x) = 1/x$ mod 1, then $\frac{1}{\log(2)} dx/(1+x)$ is an invariant measure for the Gauss map T. It is important in number theory because if $x = [a_1, ...,]$ is the continued fraction expansion of x, then $\sum_{i=0}^{n-1} f(T^i x) = \sum_{i=1}^n a_i$ if $f(x) = [1/x]$. Also this measure is known to be ergodic for T. 4) If $T(x) = 10x$ mod 1, then the Lebesgue measure dx is an invariant measure. But there are many more invariant measures: for example, the measure located on the set $\{x_1 = 0.123123123123..., x_2 = 0.23123123123..., x_3 = 312312312312...\}$ which assigns weight 1/3 to each point is an invariant measure.

LYAPUNOV EXPONENT OF INVARIANT MEASURE.
For an invariant measure μ of a continuously differentiable map f, define the **Lyapunov exponent** $h_\mu(T) = \int_0^1 \log|f'(x)| \; d\mu(x)$. If μ is supported on a periodic point, then the periodic orbit is stable if and only if the Lyapunov exponent is negative. Numerically, Lyapunov exponents can be computed by looking at the accumulation points of $\frac{1}{n} \sum_{k=1}^n \log|f'(T^k(x)|$ for large n (usually the limit exists). The reason why this works is that the measures $\mu_n(f) = \sum_{k=1}^n f(T^k x)$ have as accumulation points invariant measures.



PHYSICAL AND SRB MEASURES. In many situations, the limiting measure of $\mu_n(f) = \sum_{k=1}^n f(T^k x)$ exists and is the same for almost all initial conditions. One calles such a measure a **physical measure**. In the case of a positive Lyapunov exponent and if the measure is absolutely continuous (given by a density $\mu(f) = \int \rho(x) f(x) \; dx$), such measures are called **Sinai-Ruelle-Bowen measures** or simply **SRB-measures**. It is difficult to decide whether for a specific parameter c in the quadratic family, there is such a measure or not. One knows for example that for an open and dense set of parameters, the map has a periodic attractor with a negative Lyapunov exponent.

EXAMPLES.
1) $T(x) = 4x(1-x)$ has the Lyapunov exponent $\int_0^1 \log|4-8x|/\sqrt{x(1-x)} \; dx = \log(2)$.
2) $T(x) = \frac{1}{x}$ mod 1 has the Lyapunov exponent $\frac{1}{\log(2)} \int_0^1 -2\log(x)/(1+x) \; dx$.
3) $T(x) = 7x$ mod 1 with the invariant measure dx and the Lyapunov exponent $\log(7)$.