2/4/05 LECTURE SUMMARY **Math118, O.Knill**

Checklist of terms introduced in the first lecture:

- **Dynamical system:** defined by map $T : X \to X$ or by differential equation $\dot{x} = f(x)$.
- **Orbits:** $T^n(x) = x_n$ in discrete case and $x(t)$ in continuous case. The initial point x_0 rsp. $x(0)$ is called the **initial condition**.
- **Periodic orbits:** $T^n(x) = x$ rsp $x(t+T) = x(t)$ also called **cycles**. Examples:
  * game of life cellular automaton (demo).
  * quadratic map: $x = 3/4$ is a periodic cycle of period 1 of $T(x) = 4x(1-x)$, $x_0 = (5-\sqrt{5})/8$ is the initial condition for a periodic cycle of period 2.
  * Billiard orbits: maximal and minimal diameters of a convex table are periodic orbits.
- **Attractor:** (first lose definition): a set K such that for all $x \in X$, the orbit through x converges to K. $T^n(x) \to K$ for $n \to \infty$ or $x(t) \to K$ for $n \to \infty$. Examples:
  * Lorentz attractor (demo)
  * periodic orbits in quadratic map.
  * diagonal for square root algorithm.
- **Existence of solutions:** $x(t)$ is defined for all t, or $T^n(x)$ is defined for all x. This is not always satisfied: for example
  * $T(x) = 1 - 1/x$
  * $\frac{d}{dt}x = x^2$.

  have initial conditions which lead to blow up.
- **Integral**. A function F is called an integral if F does not change under the evolution: $F(T(x)) = F(x)$ for all x or $F(x(t)) = F(x)$ for all t. Examples:
  * $F(x,y) = xy$ for map
  $$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x+y}{2} \\ \frac{2xy}{x+y} \end{bmatrix}$$
  * $H(x,y) = x^2 + y^2$ for differential equation
  $$\begin{bmatrix} \frac{d}{dt}x \\ \frac{d}{dt}y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} .$$

An example for an application of dynamical systems: an algorithm for computing square roots:

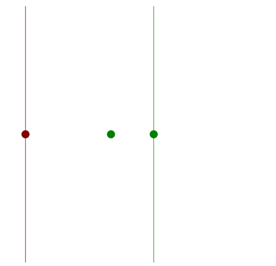
The orbit $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x+y}{2} \\ \frac{2xy}{x+y} \end{bmatrix}$ with initial condition $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ 1 \end{bmatrix}$ converges to $\begin{bmatrix} \sqrt{a} \\ \sqrt{a} \end{bmatrix}$.

Proof.

Our initial point is $(a,1) = (x_0, y_0)$. Let $T(x_0, y_0) = (x_1, y_1)$.

Key observation: The distance between x_1 and y_1 is less than half the distance between x_0 and y_0. In formulas:

$$|y_1 - x_1| \leq \frac{1}{2}|y_0 - y_0|$$



Since we can apply the same reasoning now to (x_1, y_1), we see that

$$|x_2 - y_2| \leq \frac{1}{2}|y_1 - x_1| \leq \frac{1}{4}|y_0 - x_0|$$

Continuing in the same way, we see that

$$|x_n - y_n| \leq \frac{1}{2^n}|y_0 - x_0|$$

for all n.

QED.

You see, that we not only know the convergence to the square root, we also found the rate of convergence.