INTEGRABLE MAPS IN TWO DIMENSIONS — Math118, O. Knill

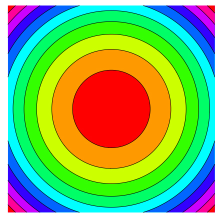
ABSTRACT. A map T in the plane is called integrable, if there is a non-constant continuous function $F(x, y)$ which is invariant under T. We give examples of integrable maps.

INTEGRABILITY. A map T is called **integrable**, if there exists a real valued continuous function $F(x, y)$ called **integral** for which the level sets $F = c$ are curves, or points and for which the identity

$$F(T(x, y)) = F(x, y)$$

holds for all (x, y).

EXAMPLES.



1. Let $T(x, y) = (\cos(\alpha)x - \sin(\alpha)y, \sin(\alpha)x + \cos(\alpha)y)$ be a rotation in the plane. It is integrable: the function $F(x, y) = x^2 + y^2$ is an integral.

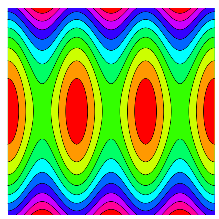
2. The map $T(x, y) = (3x, y/3)$ is integrable with integral $F(x, y) = xy$.

3. The map $T(x, y) = (x + \sin(y), y)$ is integrable with integral $F(x, y) = y$.

4. The cat map $T(x, y) = (2x + y, x + y)$ on the two dimensional torus is not integrable as you will demonstrate in a homework.

AN EXAMPLE FROM PHYSICS.

THEOREM. For every smooth function F, we can find a map, which has this function as an integral.



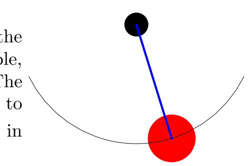
Consider the system of differential equations $\frac{d}{dt}x = F_y(x, y)$, $\frac{d}{dt}y = -F_x(x, y)$. By the chain rule, we have

$$\begin{aligned}\frac{d}{dt}F(x(t), y(t)) &= F_x(x(t), y(t))\frac{d}{dt}x(t) + F_y(x(t), y(t))\frac{d}{dt}y(t) \\ &= -\frac{d}{dt}y(t)\frac{d}{dt}x(t) + \frac{d}{dt}y(t)\frac{d}{dt}x(t)\end{aligned}$$

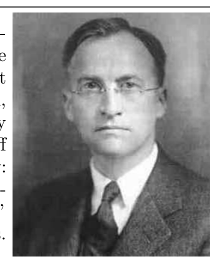
so that F does not change along a solution of the system. Define the map

$$T(x, y) = (x(1), y(1))$$

if $x = x(0), y = y(0)$. This map has F as an integral.



In physics, the function F is often called the **energy** or **Hamiltonian** of the system. The fact that F is an integral is then energy conservation. For example, for $F(x, y) = \cos(x) + y^2/2$, one obtains the energy of the **pendulum**. The differential equations are then $\frac{d}{dt}x = y$, $\frac{d}{dt}y = \sin(x)$. They are equivalent to the Newton equations $\frac{d^2}{dt^2}x = \sin(x)$. We will look at differential equations in the plane in the next week.



BIRKHOFF ON INTEGRABILITY. Like "Chaos", "Integrability" is a mathematical term, which has many different definitions. One has to specify what one means with "integrable". The fact that one has to deal with several different definitions for integrability" expressed Birkhoff in the following way "When, however, one attempts to formulate a precise definition of integrability, many possibilities appear, each with a certain intrinsic theoretical interest". Birkhoff suggested his own (as he admits not very precise) definition of integrability: there exists a finite set of periodic orbits, around which the formal series development converge and which allow to represent any solution of the system." This Birkhoff integrability is probably hard to check in a specific applications.

THE SBKP MAP. For $|k| < 1$, lets call the map

$$T(x, y) = (2x + 4 \cdot \arg(1 + k \cdot e^{-ix}) - y, x)$$

on the torus the **Suris-Bobenko-Kutz Pinkall map**. It had been found by Bobenko,Kutz, Pinkall and independently by Suris. Even so the map uses complex numbers for its definition, it is real. The argument $arg(z)$ of a complex number $z = x + iy = r\cos(\alpha) + ir\sin(\alpha) = re^{i\alpha}$ is defined as the angle α.

THEOREM. The SBKP map is integrable.

PROOF. The function

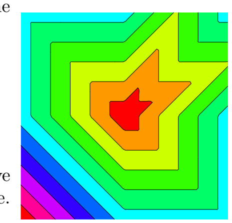
$$F(x, y) = 2(\cos(x) + \cos(y)) + k \cdot \cos(x + y) + k^{-1} \cdot \cos(x - y)$$

is an **integral**. It is not easy to verify that. Don't ask how F was found!

THE COHEN-COLINE-DE VERDIERE MAP. The map

$$T(x, y) = (\sqrt{x^2 + \epsilon^2} - y, x)$$

in the plane is called the **Cohen-Coline-de Verdiere map**. By rescaling coordinates in R^2, we can assume $\epsilon = 0$ or $\epsilon = 1$. For $\epsilon = 0$, the map has the form



$$T(x, y) = (|x| - y, x) .$$

We call it the **Knuth map**.

THEOREM (KNUTH) The Knuth map is integrable.

PROOF. We check that $T^9 = Id$. Note that the map is piecewise linear, we only have to look at the orbits of the x axes to understand the entire picture. Actually, every orbit is periodic with period $1, 3$ or 9.

LEMMA. A map in the plane for which there exists n such that $T^n(x, y) = (x, y)$, must be integrable.

PROOF. Take $f(x, y) = y$ for example. Then $F(x, y) = \sum_{k=0}^{n-1} f(T^k(x, y))$ is an integral.

If we apply this lemma to the Knuth map, we get an explicit integral

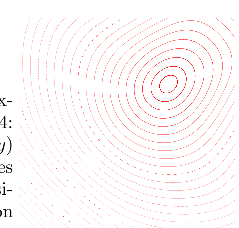
$$F(x, y) = y + |y - |x|| + |x - |y - |x||| + |y - |x - |y||| + |x - |y| + |y - |x - |y|||| .$$

The level curves of this function are shown in the graphics above. For every value $c > 0$ the level set $F(x, y) = c$ is a closed gingerman shaped curve on which T is conjugated to a rotation by an angle $1/9$.

Remark: The problem of proving periodicity of the map has been posed by Morton Brown in the American Mathematical Monthly 90, 1983, p. 569. The Monthly published the elegant solution of Donald Knuth in the volume 92, 1985 p. 218.

INTEGRAGBLE OR NOT? Lets look at the case $\epsilon = 1$, where

$$T(x, y) = (\sqrt{x^2 + 1} - y, x)$$



All orbits seem all to lie on invariant curves. The map looks integrable.
It had been communicated to me by M. Rychlik in 1998, that numerical experiments by John Hubbard revealed a hyperbolic periodic orbit of period 14: $(x, y) = (u, u)$ with $u = 1.54871181145059$. The largest eigenvalue of $dT^{14}(x, y)$ is $\lambda = 1.012275907$. The existence of a hyperbolic point of such a period makes integrability unlikely since homoclinic points might exist, but it is not impossible. It is difficult to find other hyperbolic periodic points. An other indication for non-integrability is a result of Rychlik and Torgenson who have shown that this map has no integral given by algebraic functions.

What follows was added after the handout was distributed in class:

HOW TO FIND AN INTEGRAL?

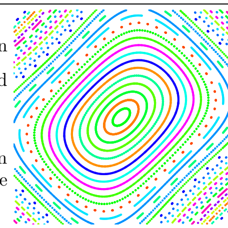
If we know a map is integrable, we could recover the invariant function F by taking $f(x,y) = y$ and defining $F(x,y) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x,y))$.

This invariant function is called the **time average** along the orbit. In the case of nonintegrability, this function is constant on complicated sets or even be infinite on some part of the plane. If the map is integrable with a nice analytic function, one could expect the integral to found using time averages.

THE McMILLAN MAP $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2kx}{(1+x^2)-y} \\ x \end{bmatrix}$ is an other example of an integrable map, where k is a parameter. It is called the McMillam map and has the integral

$$F(x,y) = x^2 + y^2 + x^2y^2 - 2kxy .$$

It is especially interesting to study because T is a rational function, a fraction of two polynomials. I don't think, one has a complete list of all integrable rational maps in the plane.



WHAT HAPPENS CLOSE TO THE INTEGRABLE CASE? In general, integrability gets lost when making small changes to an integrable map. For example, the Standard map $T(x,y) = (2x - y + \epsilon\sin(x), x)$ can for small ϵ be considered as a **perturbation** of the integrable map $T(x,y) = (2x-y,x)$ which has the integral $F(x,y) = x - y$. A study of the stable and unstable manifolds of the hyperbolic fixed point $(0,0)$ shows that they intersect transversely for small ϵ. One usually studies the map in an other form. Because $H(x,y) = (-x, y-x) = H^{-1}(x,y)$ satisfies $H(S(H(x,y)))$, where $T(x,y) = (2x - y + \epsilon\sin(x), x)$ and $S(x,y) = (x + y + \epsilon\sin(x), y + \epsilon\sin(x))$, we can look also at the map S instead. This map has the integral $F(x,y) = y$ for $\epsilon = 0$ and the invariant curves are horizontal.

KAM. Near integrable maps, remnants of integrability still exist. These traces of integrability persist in the form of smooth **invariant curves** which are now called KAM curves. The acronym KAM stands for Kolmogorov-Arnold-Moser. The proof that invariant curves persist after the perturbation is not easy. To find an invariant curve on which the map is conjugated to an irrational rotation with angle α, we need to find a periodic function $q(x)$ such that $q_n = q(n\alpha)$ satisfies the nonlinear recursion $q_{n+1} - 2q_n + q_{n-1} = \epsilon\sin(q_n)$. This means

$$q(x+\alpha) - 2q(x) + q(x) = \epsilon\sin(q) .$$

Naively, one could try to find q using the **implicit function theorem**: if one could invert the linear map $L(q) = q(x+\alpha) - 2q(x) + q(x)$.

SMALL DIVISORS. Lets look at this inversion problem If $q(x) = \sum_n c_n e^{inx}$ is the Fourier series of q, then $Lq(x) = \sum_n c_n(e^{i\alpha} - 2 + e^{-i\alpha})e^{inx}$. If $L(q) = p = \sum_n d_n e^{inx}$. then

$$q = L^{-1}p = \sum_n d_n \frac{1}{e^{in\alpha} - 2 + e^{-in\alpha}} e^{inx} = \sum_n d_n \frac{2}{\cos(n\alpha) - 1} e^{inx} .$$

You see the appearance of **small divisors** $\frac{2}{\cos(n\alpha)-1}$. In order that the Fourier series of the inverse converges, one needs α to be far away from rational numbers. Such numbers are called **Diophantine numbers**. Evenso, one is able to invert L in certain cases, the map L is not invertible as required for the implicit function theorem. One needs a so called **hard implicit function theorem**.