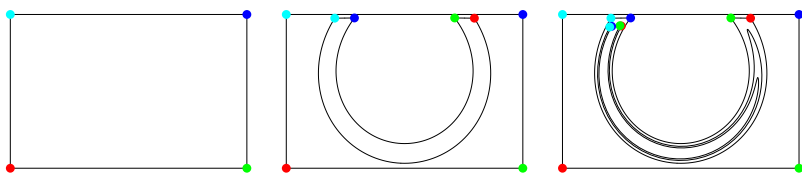


HORSE SHOES AND HOMOCLINIC TANGLE

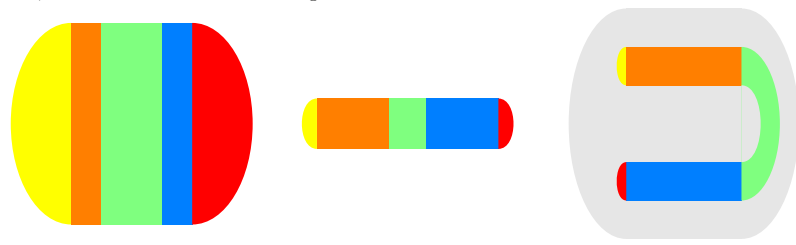
Math118, O. Knill

ABSTRACT. The horse shoe map is a map in the plane with complicated dynamics. The horse shoe construction applies also to conservative maps and occurs near homoclinic points.

THE HORSE SHOE MAP. We construct a map T on the plane which maps a rectangle into a horse-shoe-shaped set within the old rectangle. The following pictures show an actual implementation with an explicit map



A better picture, which can be found in the book of Gleick shows a now rounded region which is first stretched out, then bent back into the same region.



THE HORSE SHOE ATTRACTOR. The map T maps the region G into $T(G)$ which is a subset of G . The image $T(T(G))$ is then a subset of $T(G)$ etc. The intersection K^+ of all the sets $T^n(G)$ is called the **horse shoe attractor**. It is invariant under the map T but T is not invertible on K^+ . But now look at the set of points K which do not leave the original rectangle when applying T^{-1} . This set is now T invariant.

THEOREM (Smale) The map T restricted to K is conjugated to the shift map $S(x)_n = x_{n+1}$ on the space X of all 0-1 sequences. With a suitable distance function defined on X , the conjugating map is continuous, invertible and its inverse is also continuous.

Similar as with the tent resp. Ulam map, we will prove this only later when we cover the shift map. The conjugation is a coding map: for a point $z \in K$, define the sequence x_n as follows. Call C_0 the left of the three rectangles and C_1 the right one.

$x_n = \begin{cases} 1, & \text{if } T^{-n}(z) \in C_1 \\ 0 & \text{if } T^{-n}(z) \in C_0 \end{cases}$. It has to be shown that every point $z \in K$ is associated to exactly one 0-1 sequence.

HORSE SHOES IN REAL MAPS. The horse shoe map often occur in an iterate of maps. One can see this sometimes directly. In the picture to the right, we iterated the points in a disc using the Standard map

$$T(x, y) = (2x + y + c \sin(x), x)$$

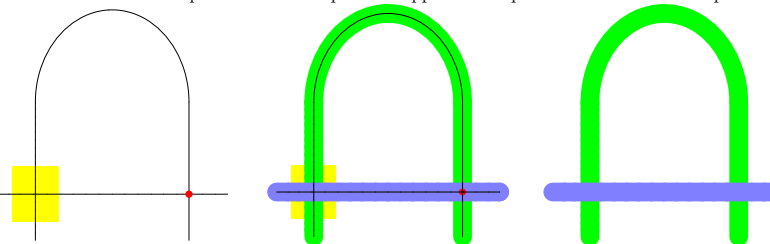
on the torus. The picture has been made with the parameter value $c = 2.4$. We took a disc and applied the map T 5 times.



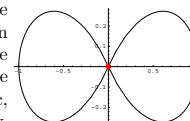
HORSESHOE FROM HOMOCLINIC POINTS.

THEOREM. A transverse homoclinic point leads to a horseshoe. There exists then an invariant set, on which the map is conjugated to a shift on two symbols.

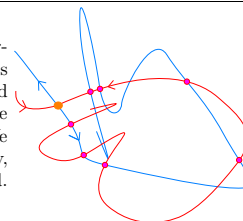
Take a small rectangle A centered at the hyperbolic fixed point. Some iterate of T will have the property that $T^n(A)$ contains the homoclinic point. Some iterate of the inverse of T will have the property that $B = T^m(A)$ contains the homoclinic point to. The map T^{n+m} applied to B produces a horseshoe map.



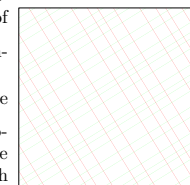
MANY HYPERBOLIC PERIODIC POINTS. The conjugation shows that periodic points are dense in K and that it contains dense periodic orbits. The map T restricted to K is chaotic in the sense of Devaney. Actually, one can show that each of the periodic points in K form hyperbolic points again. The stable and unstable manifolds of these hyperbolic points form again transverse homoclinic points and the story repeats again. This story is pretty generic, but there are cases, where stable and unstable manifolds come together nicely. This must be the case in integrable systems.



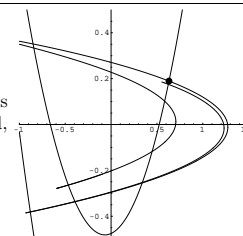
THE HOMOCLINIC TANGLE. If stable and unstable manifolds of a hyperbolic fixed point intersect, then they must intersect a lot more. The reason is that the image of the intersection produces another intersection of stable and unstable manifolds because both curves are invariant. Note however that the stable manifold can not intersect with itself, except at hyperbolic points. We have in general two curves in the plane, both of which wind around like crazy, produce a lot of hyperbolic points due to all the horse-shoes which are created.



CAT MAP For the Cat map, one can compute the stable and unstable manifolds explicitly. The point $(0,0)$ is a fixed point. The Jacobean matrix of $T(x, y) = (2x + y, x + y)$ is $T'(x, y) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. The eigenvector to the eigenvalue $(3 + \sqrt{5})/2$ is $\begin{bmatrix} (1 + \sqrt{5})/2 \\ 1 \end{bmatrix}$. The eigenspace is a line. It is the unstable manifold. When wrapped around the torus and plotted on the square, it appears as an infinite sequence of parallel line segments. Similarly, the stable manifold is a curve which winds around indefinitely around the torus. Each intersection of these lines is a homoclinic point.



HENON MAP In the case of the Henon map, stable and unstable manifolds intersect transversely in general. The map has all the complexity described, and especially, infinitely many hyperbolic points.



The notes below were added after the notes were distributed in class.

NICE INTEGRALS. Let us call smooth function $F(x, y)$ **nice** if any intersection of the set $F(x, y) = c$ with a bounded rectangle consists of a finite union of curves and points only. Let us call a map **nicely integrable** if it has a nice integral.

THEOREM (Poincare). A map with a transverse homoclinic point can not be nicely integrable.

PROOF. If T is nicely integrable, then also each iterate T^n is nicely integrable. The invariant horse shoe of some iterate of T is a set in which each point is accumulated by other points of the set. The horse shoe set can not be contained in a finite union of curves. Since the invariant function F must be constant on the horse shoe, the function can not be nice. The level set either contains infinitely many points or infinitely many curves in a rectangle which contains the horse shoe.

(Poincare knew this result only for analytic integrals).