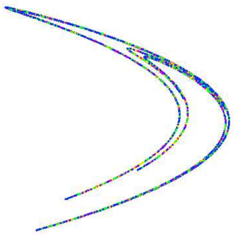
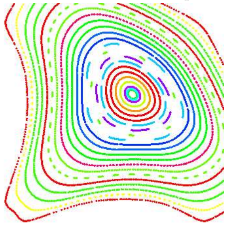
2/14/2005: MAPS IN TWO DIMENSIONS Math118, O. Knill

ABSTRACT. In this first lecture, we look at the dynamics of maps in the plane and introduce some terminology related to the Jacobian matrix $DT(x,y)$.

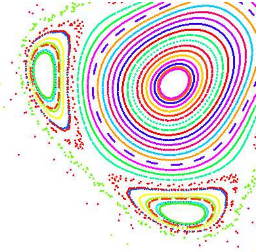
THE HENON MAP. The map $T(x,y) = (x^2 + c - by, x)$ with parameter b, c is called the **Henon map**. For $b = 0$, the map can be understood by a the one-dimensional quadratic map $f(x) = x^2 + c$.



Orbits of $T(x,y) = (-1.5x^2 - 0.3y, x)$ accumulate on an attractor.



Oribits of the map $T(x,y) = (-0.5x^2 + 1 - y, x)$.



Oribits of the map $T(x,y) = (0.4x^2 + 1 - y, x)$.

THE JACOBEAN. Two smooth functions $f(x,y)$ and $g(x,y)$ of two variables, define a map

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$

in the plane. We say, a map is **area preserving** or **conservative** if $T(A)$ has the same area then A for any rectangle A. We write partial derivatives as $f_x(x,y), f_y(x,y)$.

THEOREM. T is area-preserving if and only if the determinant of the Jacobian matrix $DT(x,y) = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix}$ is equal to 1 or -1 at all points (x,y).

Proof. This is the change of variable formula in multi-variable calculus. An elegant way to verify the formula is to interpret the map to define a parameterization of a surface $(u,v) \to \vec{r}(u,v) = (f(u,v), g(u,v), 0)$ which is also called the uv-map. You know the surface area element as $|\vec{r}_u \times \vec{r}_v|$ which is $|f_x g_y - g_x f_y| = |\det(DT(x,y))|$.

If $|\det(DT(x,y))| < 1$ everywhere, then the map is called **dissipative**. It shrinks volume. If $\det(DT(x,y))| > 0$ for all (x,y), the map is called **orientation preserving**.

EXAMPLE. The Henon map $T(x,y) = (x^2 + c - by, x)$ is area preserving if and only if $|b| = 1$. and orientation preserving for positive b because the Jacobian matrix is

$$DT(x,y) = \begin{bmatrix} 2x & -b \\ 1 & 0 \end{bmatrix}$$

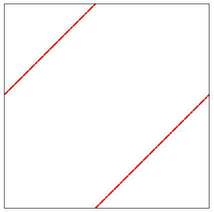
has the determinant $\det(DT) = b$.

SECOND ORDER DIFFERENCE EQUATIONS. A recursion like $x_{n+1} = x_{n-1} + \sin(x_n)$ defines a map if we introduce $y_n = x_{n-1}$:

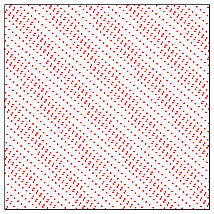
$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} y_n + \sin(x_n) \\ x_n \end{bmatrix} .$$

You might have seen the **Fibonacci recursion** $x_{n+1} = x_n + x_{n-1}$.

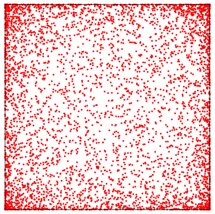
PRODUCT OF 1D MAPS. If f, g are one dimensional maps, then $T(x,y) = (f(x), g(y))$ is the product of these maps. The orbits of T is determined by the orbits of f and g. We just run the two dynamical systems in parallel. Examples:



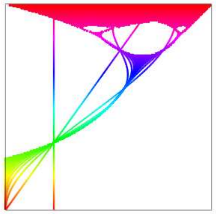
$T(x,y) = (x + \alpha, y + \alpha)$. The map is area preserving. $DT(x,y)$ is the identity matrix. Orbits are on line of slope 1.



$T(x,y) = (x + \alpha, y + \beta)$. Again $j(x,y)$ is the identity matrix. Orbits can be dense. Also this map is area preserving.



$T(x,y) = (4x(1-x), 4y(1-y))$ is not conservative. $\det(DT(x,y)) = 16(1-2x)(1-2y)$.

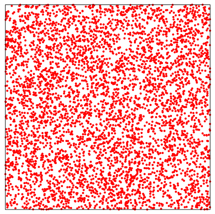


$T(x,y) = (4yx(1-x), y)$. This map is not area-preserving.

THE CAT MAP. If A is a matrix with integer entries, then $T\vec{x} = A\vec{x}$ defines a map on the **torus** R^2/Z^2, which means we take x mod 1 and y mod 1. The example

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ x + y \end{bmatrix}$$

is called the "cat map". Arnold had illustrated the map using a cat. It belongs to a class of dynamical systems which can be understood completely. They are extremely "chaotic".



An orbit of the "cat map".

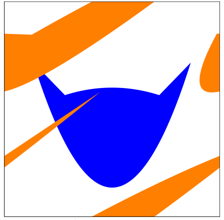


Image of the "cat" on the torus.

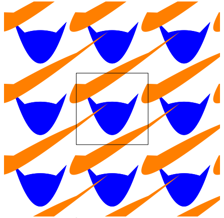
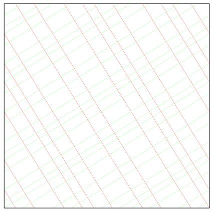
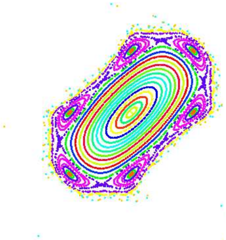


Image of the "cat" in the plane.

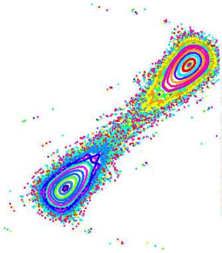


Invariant directions.

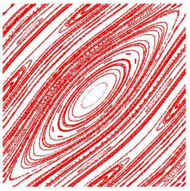
CUBIC HENON MAP AND THE STANDARD MAP. The map $T(x,y) = (cx - x^3 - y, x)$ with parameter c is called **cubic Henon map**. It is area preserving. The map $T(x,y) = (2x + c\sin(x) - y, x)$ is a map on the torus called the **Chirikov Standard map**. It is area preserving for all parameters c.



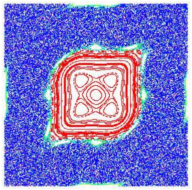
Orbits of $T(x,y) = (cx - x^3 - y, x)$ for $c = 1.5$.



Orbits of $T(x,y) = (cx - x^3 - y, x)$ for $c = 2.5$.



Orbits of $T(x,y) = (2x + c\sin(x) - y, x)$ for $c = 0.5$.



Orbits of $T(x,y) = (2x + c\sin(x) - y, x)$ for $c = 2.1$.