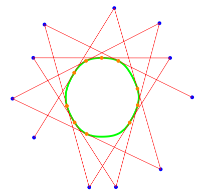
EXTERIOR BILLIARDS — Math118, O. Knill

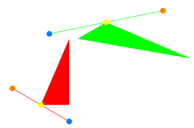
ABSTRACT. We look here briefly at the dynamical system called "exterior billiard". Affine equivalent tables lead to conjugated dynamical systems. One does not know, whether there is a table for which an orbit can escape to infinity nor does not know whether the ellipse is the only smooth convex exterior billiard table for which the dynamics is integrable.

EXTERIOR BILLIARDS. Dual billiards or **exterior billiards** is played outside a convex table γ. Take a point (x, y) outside the table, form the tangent at the table and reflect it at the tangent point (or the mid-point of the interval of intersection). To have no ambiguity with the tangent, γ is oriented counter clockwise. The positive tangent is the tangent at the curve in the same direction.

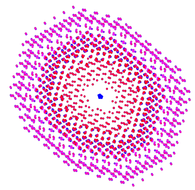


EQUIVALENCE. Assume $S(x) = Ax + v$ is an affine transformation in the plane, where A is a linear transformation and v is a translation vector. Given two tables γ_1, γ_2 such that $S(\gamma_1) = \gamma_2$, then the exterior billiard systems T_{γ_1} and T_{γ_2} are conjugated.

PROOF. Unlike angles, affine transformations preserve ratios and a trajectory of the the exterior billiard at γ_1 is mapped into a trajectory of the table γ_2.

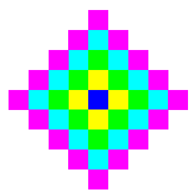


EXAMPLE POLYGONS. Already the case of polygons can be complex. Exterior billiard at a general quadrilateral (=four sided polygon) shows already interesting dynamics. Note that the exterior billiard map is not continuous for polygons. One already does not know whether orbits stay bounded for all quadrilaterals. For regular pentagons, Tabatchnikov was able to compute the Hausdorff dimensions of the closure of some orbits. They are fractals.

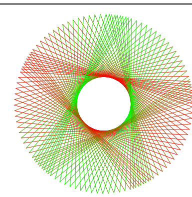


INTEGRABLE PARALLELEPIPED. The exterior billiard at a parallelepiped is integrable.

PROOF. By affine equivalence, it is enough to show this for squares. Check that every orbit is periodic.

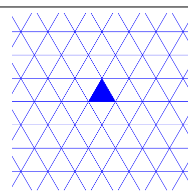


INTEGRABLE ELLIPSE. The exterior billiard at an ellipse is integrable.
PROOF. By affine equivalence, it is enough to show this for circles.



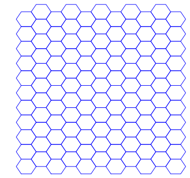
INTEGRABLE TRIANGULAR BILLIARD. The exterior billiard at any triangle is integrable.

PROOF. By affine equivalence, it is enough to show integrability for equilateral triangles. Since every orbit is periodic, we have integrability by a lemma proven earlier. For the hexagon, we have also the property that every orbit is periodic.

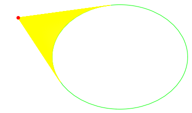


INTEGRABLE HEXAGONAL BILLIARD. The exterior at a regular hexagon is integrable.

PROOF. The key is to see that the successive reflections of the sides of the polygon at the corners of the polygon produces a regular tessellation of the plane.



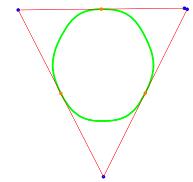
GENERATING FUNCTION. Similar as for billiards, there is a generating function $h(x, x')$ for the exterior billiard. Given two polar angles ϕ, ϕ', draw the tangents with this angle. The function $h(\phi, \phi')$ is the area of the region enclosed by these lines and the curve. We can check that the partial derivative $\frac{\partial}{\partial x} h(\phi, \phi') = -r^2/2$, where r is the distance from the point to the point of tangency. The exterior billiard is area-preserving.



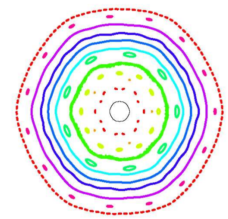
PERIODIC POINTS. By maximizing the functional

$$H(x_1, ..., x_n) = \sum_{k=1}^{n} h(x_i, x_{i+1})$$

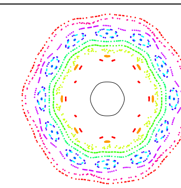
one obtains periodic orbits of the exterior billiard. To say it in words: among all closed polygons for which all sides are tangent to the table, the ones which maximizes the sum of the areas $h(x_i, x_{i+1}$ form a periodic orbit of the dual billiard.



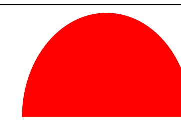
INVARIANT CURVES. For smooth tables, every orbit is bounded. This is a consequence of KAM (Kolmogorov-Arnold-Moser) theory. In that case, there are invariant curves far from the table which enclose the table. A point on this curve will remain on this curve for all times and the dynamics is conjugated to a Kronecker system. A proof of the "invariant curve theorem" is not easy: it requires heavy analytic artillery, modifications of the Newton method or "hard" implicit function theorems. One has to find a smooth invertible map on the circle such that $h(q(x - \alpha), x) + h(x, q(x + \alpha)) = 0$ is satisfied. The irrational rotation number α has to be "far away from rational numbers", one calls this Diophantine. For the story of dual billiards, the proof is even more tricky and has been done by R. Douady.



AN UNSOLVED PROBLEM. Is the ellipse the only smooth convex table for which exterior billiard is integrable?



AN UNSOLVED PROBLEM. Is there a table with an unbounded orbit?
An example of where one does not know the answer, is a semicircle. Tabatchnikov states numerical evidence that for this billiard, there is an unbounded orbit.



HISTORY.
1960. The problem is suggested by B.H. Neumann
1963 The problem is posed by P.Hammer in a list of unsolved problems
1973 In Moser's book "Stable and Random Motion", the stability problem is raised. Some people call exterior billiard also the Moser billiard.
1978 The exterior billiard is also featured in Mosers Intelligencer article "Is the solar system stable".
The photo of Moser to the right had been taken by J. Pöschel in the year 1999, when Moser was lecturing in Edinburgh about twist maps. Moser died in the same year.