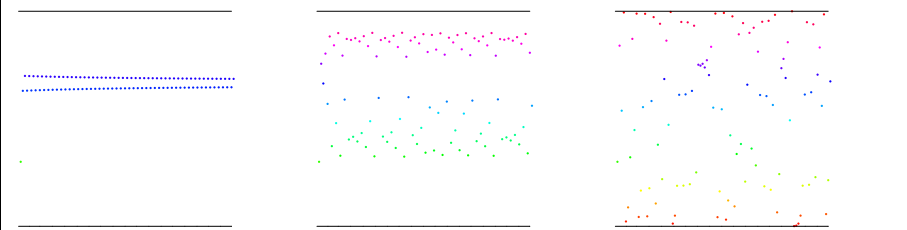


2/4/04 EXAMPLES OF DYNAMICAL SYSTEMS Math118, O.Knill

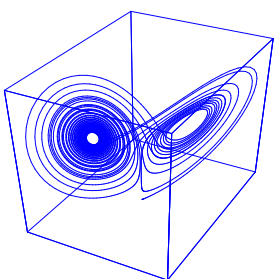
ABSTRACT. In this lecture, we look at examples of dynamical systems. Most examples in this zoo of systems belong to the "hall of fame". They are "stars" in the world of all dynamical systems and will appear later in this course.

THE LOGISTIC MAP. $T(x) = cx(1 - x)$. This is an example of an **interval map**. The **parameter** c is fixed in the interval $[0, 4]$. Lets look at some **orbits**. To compute an orbit say for $c = 3.0$, start with some **initial condition** like $x_0 = 0.3$, and iterate the map $x_1 = T(x_0) = 3x_0(1 - x_0) = 0.63$, $x_2 = T(x_1) = 2x_1(1 - x_1) = 0.6993$ etc. Lets do this with the computer. We show a few orbits for different parameters c . We always start with the initial condition $x_0 = 0.3$. Time is the horizontal axes and the interval $[0, 1]$ is on the vertical axes.



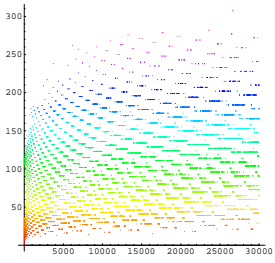
THE LORENTZ SYSTEM. The system of differential equations

$$\begin{aligned} \dot{x} &= 10(y - x) \\ \dot{y} &= -xz + 28x - y \\ \dot{z} &= xy - \frac{8z}{3} \end{aligned}$$



is called the **Lorentz system**. We see a numerically integrated orbit $(x(t), y(t), z(t))$ which is attracted by a set called the **Lorentz attractor**. It is an example of what one calls a **strange attractor**. Orbits behave chaotically on that set in the sense that one observes sensitive dependence on initial conditions. The set is also measured to be a fractal, of dimension strictly between 1 and 2.

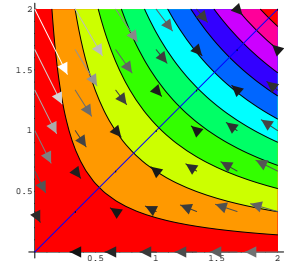
THE COLLATZ PROBLEM. Define a map T on the positive integers as follows. If n is even, then define $T(n) = n/2$, if n is odd, then define $T(n) = 3n + 1$. One believes that every orbit $n, T(n), T(T(n))$ will end up at 1 but one does not have a proof and there are people who think that mathematics is not ready for this problem. Theoretically, it would be possible that an orbit escapes to infinity, or that there exists a **periodic orbit** $n, T(n), T^2(n), \dots, T^k(n) = n$. The problem is also called **Ulam problem** or $3n + 1$ **problem**. It is a notorious open problem. The picture to the right shows how long it takes to get from n to 1.



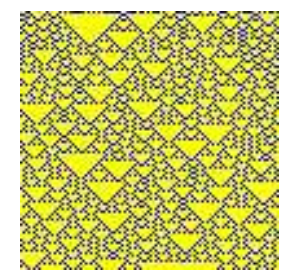
COMPUTING SQUARE ROOTS. Look at the map

$$T(x, y) = \left(\frac{2xy}{x+y}, \frac{x+y}{2} \right)$$

which assigns to two numbers a new pair, the **harmonic means** as well as the **algebraic mean**. You can easily check that the quantity $F(x, y) = xy$ is preserved: $F(T(x, y)) = F(x, y)$. It is called an **integral**. A map in the plane with such an integral is called an **integrable system**. All orbits converge to the line $x = y$ which consists of **fixed points**. Why is this useful? Start with $(1, 5)$ for example. The sequence (x_n, y_n) will converge to the diagonal and so to $(\sqrt{5}, \sqrt{5})$. Lets do it: we have $(1, 5), (\frac{5}{3}, 3), (\frac{15}{7}, \frac{7}{3}), (\frac{105}{47}, \frac{47}{21})$. We know that $\sqrt{5}$ is in the interval $[x_n, y_n]$ for all n . For example, $47/21 = 2.238\dots$ is already a good approximation to $\sqrt{5} = 2.236\dots$



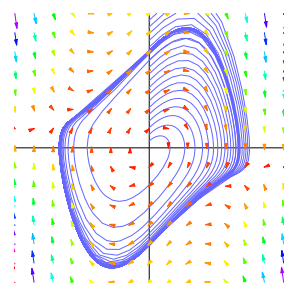
CELLULAR AUTOMATA. Given a infinite sequence x of 0's and 1's, define a new sequence $y = T(x)$, where each entry y_n depends only on x_{n-1}, x_n, x_{n+1} . There are 256 different automata of this type. The picture below shows an orbit of "Rule 18". One of the interesting features of this automaton is that its evolution is linear on parts of the phase space. The nonlinear and interesting behavior is the motion of the **kinks**, the boundaries between regions with linear motion. A sequence x has a kink at n , if for some $k \geq 0$, $[x_{n-k}, \dots, x_{n+k+1}] = [1, 0, \dots, 0, 1]$, like the pattern 10001.



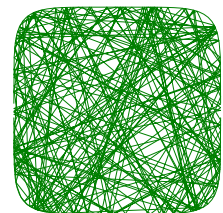
DIFFERENTIAL EQUATIONS IN THE PLANE. Second order differential equations can be written as differential equations in the plane. An example is the **van der Pool oscillator**

$$\begin{aligned} \frac{d}{dt}x &= y \\ \frac{d}{dt}y &= -x - (x^2 - 1)y; \end{aligned}$$

which shows a **limit cycle**. All orbits (except with the initial condition $(0, 0)$) converge to that limit cycle.



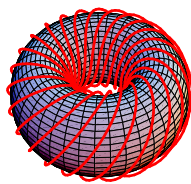
BILLIARDS. Let us take a table like the region $x^6 + y^6 \leq 1$. A ball reflects at the boundary. What is the long time behavior of this system? Is it possible that the angles a light ray makes with the boundary of the table become arbitrarily close to 0 and arbitrarily close to 180 degrees? Are there paths which come arbitrarily close to any point? The billiard flow defines a smooth map on the annulus. The study of this system has relations with elementary differential geometry. For example, the curvature of the boundary plays a role. The study of billiards is also part of a mathematical field called **calculus of variations** which deals with finding extrema of functions.



STANDARD MAP. The map

$$T(x, y) = (2x + \gamma \sin(x) - y, x)$$

on the plane is called the **Standard map**. Because $T(x + 2\pi, y) = T(x, y + 2\pi) = T(x, y)$, one can take both variables x, y modulo 2π and obtain a map on the **torus**. The real number γ is a parameter. The map appeared around 1960 in relation with the dynamics of electrons in microtrons and was first studied numerically by Taylor in 1968 and by Chirikov in 1969. The map can be completely analyzed for $\gamma = 0$. The map exhibits more and more "chaos" as γ increases. The picture to the right shows a few orbits in the case $\gamma = 1.3$.

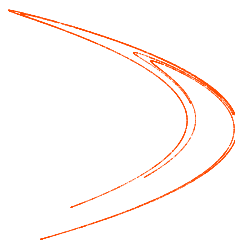


GEODESIC FLOW. Light on a surface takes the shortest possible path. These paths are called **geodesics**. On the plane, the geodesics are lines, on the round sphere, the geodesics are **great circles**, on a flat torus (see picture), the geodesics are lines too, but they wind around the surface. On some surfaces like surfaces of revolution or the ellipsoid, the geodesic flow can be analyzed completely on. On other surfaces, the flow can become very complicated. There are bumpy spheres on which each geodesic path is dense in the sense that the curve comes close to every point and also every direction at that point.

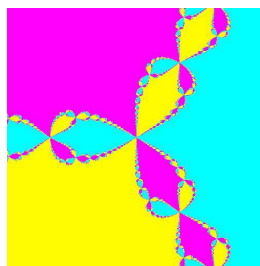
THE HENON MAP. One of the simplest nonlinear nonlinear maps on the plane is the **Henon map**

$$T(x, y) = (ax^2 + 1 - by, x) .$$

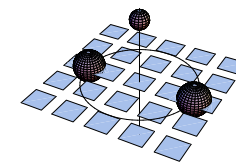
For $|b| = 1$ the map is area-preserving. For $|b| < 1$, it contracts area and produces **attractors**. The **Henon attractor** is obtained for $a = -1.4, b = -0.3$. The Henon map is equivalent to the **nonlinear recursion** $x_{n+1} = ax_n^2 + 1 - bx_n$. While **linear recursions** like the Fibonacci recursion $x_{n+1} = x_n + x_{n-1}$ can be solved explicitly using linear algebra, nonlinear recursions do no more lead to explicit formulas for x_n .



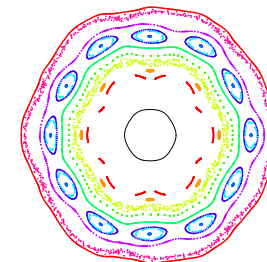
SOLVING EQUATIONS. To solve the equation $f(x) = 0$ numerically, one can start with an approximation x_0 , then apply the map the **Newton iteration map** $T(x) = x - f(x)/f'(x)$. If $T(x) = x$, then $f(x) = 0$. As long as the root y satisfies $f'(y) \neq 0$, this algorithm works for x_0 near y . The method also works in the complex. In the case of several roots, is an interesting question to explore the **basin of attraction** of a root. The picture to the right shows this in the case of $f(z) = z^3 - 2$, where one has 3 roots. Depending on the initial point z_0 , one ends up on one of the three roots. The Newton map for polynomials f defines a **rational map**. Its study is part of **complex dynamics**.



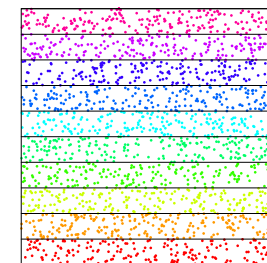
THREE BODY PROBLEM. Celestial mechanics determines very much the timing of our lives. Our calendar is based on it. While the motion of 2 bodies is understood well since Kepler, the **three body problem** is very complicated. Part of modern Mathematics, like topology have been developed in order to understand it. The **Sitnikov problem** is a **restricted three body problem** where the motion of a planet moves with negligible mass in a binary star system. The two stars circle each other on ellipses. The planet moves on the line through the center of mass, perpendicular to the plane in which the stars are located. For this system, there is a mathematical proof of some chaotic motion.



EXTERIOR BILLIARDS. A geometrically defined dynamical system has been used to capture the main difficulties of the three body problem. The system is defined by a convex table as in billiards but this time, the a point outside the table is reflected at the table boundary: take the tangent to the table in the anti-clockwise direction and take the other point which has the same distance to the touching point. The map defined on the exterior of the table is area preserving and in general very complicated. It is not known, whether there exists a table for which there are unbounded orbits.



THE DIGITS OF PI. The digits of the number $\pi = 3.14159265358979323846264338327950288419716939937510...$ appear random. With $T(x) = 10x \bmod 1$ and $f(x) = [10x]$, where $[x]$ is the integer part of a number, the number $f(T^n(x))$ is the n 'th digit of π . It appears that every digit appears with the same frequency and also all combinations of digit sequences. It is an open problem, whether this is true. One would call π normal. The picture to the right shows the sequence $x_n = f(T^n(x))$.



LATTICE POINTS NEAR GRAPHS. Given the graph of a function f on the real line, one can look at the distances to the nearest lattice points. This defines a sequence of numbers which can be generated by a dynamical system. For polynomials of degree n , the system is a map on the n dimensional torus. For the parabola $f(x) = ax^2 + bx + c$ we obtain which leads to a map of the type $T\left(\begin{matrix} x \\ y \end{matrix}\right) = \begin{bmatrix} x + \alpha \\ x + y \end{bmatrix}$ on the two dimensional torus.

