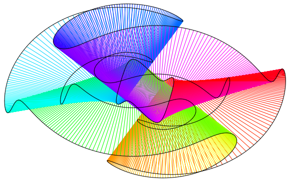
WAVE FRONTS AND CAUSTICS

Math118, O. Knill

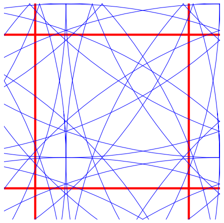
ABSTRACT. Wave fronts which start at a point evolve and break at caustics. Given a metric in Euclidean space, the wave fronts form a one-parameter family of piecewise smooth surfaces.

WAVE FRONTS AND CAUSTICS. The set of points reached at time t from a given point x form the **wave front** $K_x(t)$ of x. If the geodesics starts with an initial velocity $(\cos(\phi), \sin(\phi)$, it reaches at time t the point $K_x(t,\phi)$. A **conjugate point** of x is a point $K_x(t,\phi)$, for which $DK_x(t,\phi)$ has zero determinant. The set C_x of all conjugate points $K(t,\phi)$ form the **caustic** of x. The caustic of a curve $\phi \mapsto r(\phi)$ in the plane is defined as the set K_γ of points for which $DK_\gamma(t,\phi)$ has zero determinant, where $K_\gamma(t,\phi)$ is the point reached when we start at $r(\phi)$ in the normal direction $n(\phi)$. Given a closed compact surface and a point P. How does the wave front $K_(t)$ look like? Does it become dense on the surface?



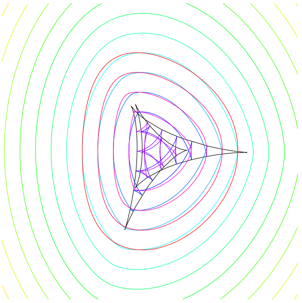
EXAMPLES.

FLAT TORUS. On the flat torus, the wave front $K_x(t)$ becomes dense on the surface for every point x. The caustic is empty. The picture to the right shows the wave front on the flat torus at time 3.



ROUND SPHERE. The wave front $K_x(t)$ is a circle or a point at all times. In the case of the flat torus, the caustic is empty, in the case of the sphere, the caustic C_x consists of two points, x and the antipole $S(x)$ of x.

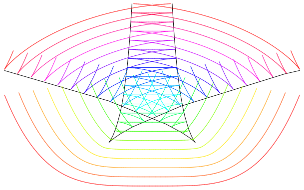
CAUSTIC FLAT CASE. Let $\gamma : r(\phi) = (x(t), y(t)$ be a curve in the flat plane and let $n(\phi) = (-y'(\phi), x'(\phi)$ be the normal vector to the curve and $\rho(\phi) = 1/||n(\phi)|| = 1/||r'||$. Then $K_\gamma(t,\phi) = r(\phi) + tn(\phi)\rho(\phi) = (x(\phi) - ty(\phi)\rho(\phi, y(\phi) + tx(\phi)\rho(\phi)$ so that $DK_\gamma(t,\phi) = \begin{bmatrix} n(\phi)\rho(\phi) & r'(\phi) + tn'(\phi)\rho(\phi) + tn(\phi)\rho'(\phi) \end{bmatrix} = \begin{bmatrix} n(\phi)\rho(\phi) & r'(\phi) + tn'(\phi)\rho(\phi) \end{bmatrix} = 1/\rho + t\rho^2(\phi) \begin{bmatrix} n(\phi) & n'(\phi) \end{bmatrix}$ using $\det(\vec{a}, \vec{b} + \vec{a}) = \det(\vec{a}, \vec{b})$. The caustic of the curve γ is called the **evolute** of the curve.



EXAMPLE: Locally, we can represent a plane curve as a graph $(x, f(x))$. The wave front $W(t,x) = (x, f(x)) + t(-f'(x), 1)/\sqrt{1 + f'(x)^2}$ has the caustic

$$\{(t,x) = (1 + f'(x)^2)^{3/2}/f''(x), x)\ \} \ .$$

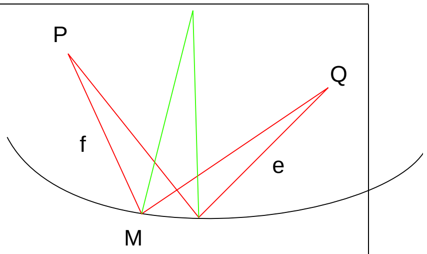
For example, for $f(x) = x^2$, we have $\{W((1 + 4x^2)^{3/2}/2, x)\} = \{(-4x^3, 1/2 + 3x^2)\ \}$ which is essentially the graph of $y = x^{2/3}$. For $f(x) = x^4$, we have $\{W((1+16x^6)^{3/2}/(12x^2), x)\ \} = \{(2x/3 - 16x^7/3, 7x^4/3 + x^{-2}/12)\ \}$.



THE MIRROR EQUATION. If P and Q are successive points on a caustic for a geodesic ray which is reflected at the boundary point M with curvature κ and impact angle θ, then $f = |P - M|$ and $e = |Q - M|$ satisfy

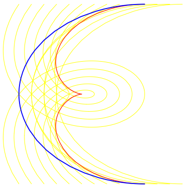
$$\boxed{\frac{1}{f} + \frac{1}{e} = \frac{2\kappa}{\sin(\theta)}}$$

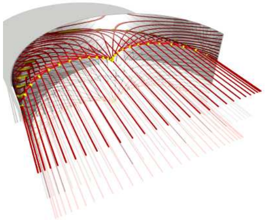
PROOF. The change of the incoming angle $d\theta_1$a and the outgoing ray $d\theta_2$ is related by $d\theta_2 = 2d\theta - d\theta_1$. The claim follows from $d\theta = 1/\rho = \kappa, d\theta_1 = \sin(\theta)/f, d\theta_2 = \sin(\theta)/e$.

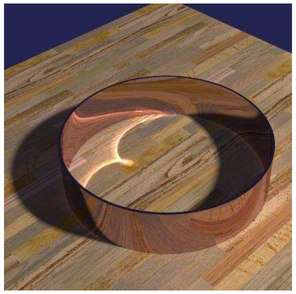


Interpretation: If $P = x$ is a point, then Q is a point of the differential geometrical caustic C_x of the point x. If you light a flashlight at P, then the point Q will be a focal point, where the light density is strong.

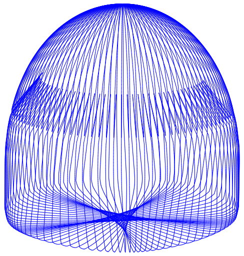
THE COFFEE CUP CAUSTIC. If $r(t) = (-\sin(t), \cos(t))$ is the boundary of the cup and light enters in the direction $(-1, 0)$, then the impact angle θ is just t. The curvature $\kappa(t)$ is 1. Parallel light coming from the right focuses at infinity so that $1/f = 0$. The light which leaves into the direction $(\cos(2t), \sin(2t))$ focuses after reflection at a distance $e = \sin(\theta)/(2\kappa) = \sin(\theta)/2$. The caustic is therefore parameterized by $(-\sin(t), \cos(t)) - (\cos(2t), \sin(2t))\sin(t)/2 = (-\sin(t) + \cos(2t)\sin(t)/2, \cos(t) + \sin(2t)\sin(t)/2)$. Image credit for the picture to the right: Henrik Wann Jensen 1996.







CAUSTICS OF BILLIARDS. The word "caustic" has different meaning in billiards and in differential geometry. Caustics can be defined for any family of light rays. In differential geometry, one looks at all the light rays which are emitted at one spot or all light rays emitted orthogonally to a given curve. If we look at all the light rays emitted from a point x in a billiard table, we will see caustics too. The differential geometrical C_x will be dense however in general. In billiards, we have looked at the caustic of a family of rays which correspond to billiard trajectories on an invariant curve. However, there are some cases, where there is a direct connection between differential geometrical caustics and caustics of billiards. We can deform a sphere in such a way that the caustic of a point on the sphere is the caustic of a special billiard table. We have used this construction once to find metrics on spheres for which the caustics is nowhere differentiable.



CAUSTICS OF BILLIARDS. Caustics of billiards can be quite complicated. To the right, we see some examples for billiards in tables of equal thickness.