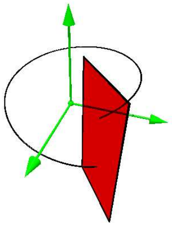
DIFFERENTIAL EQUATIONS IN THREE DIMENSIONS Math118, O. Knill

ABSTRACT. Differential equations in space can exhibit more complicated behavior than in the plane. Higher-dimensional systems occur naturally as we will see. Many systems can be studied using a Poincare map.

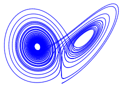
HOW DO SYSTEMS APPEAR IN THREE DIMENSIONS?

- A second order differential equation $\ddot{x} = f(x, \dot{x}, t)$ can be written with $(x, y, z) = (x, \dot{x}, t)$ as $(\dot{x}, \dot{y}, \dot{z}) = (y, f(x, z), 1)$. Such systems often appear in physcis. The time dependence allows to write the equation in three dimensions.
- A mechanical system of two degrees of freedom defines a flow in four dimensional space. Every coordinate has a position and velocity. Because energy is preserved, the dynamics takes place on a three dimensional energy surface.

POINCARE MAP. Assume we have a differential equation $\frac{d}{dt}x = F(x)$ in space. Given a two-dimensional surface Σ in space, we can start at a point in the plane, wait until the orbit returns back to the plane, hitting it transversely and so define a map from a subset of the plane to the plane. For any surface Σ in space, there is an open subset U, on which the return map T is defined and smooth.



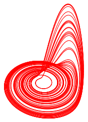
THE LORENTZ SYSTEM. The system has been suggested by Eduard Lorentz in 1963. It is obtained by a truncation of the **Navier Stokes equations**. It gives an approximate description of a horizontal fluid layer heated from below which is itself a model for the earths atmosphere.



$$\begin{aligned} \dot{x} &= a(y - x) \\ \dot{y} &= cx - xz - y \\ \dot{z} &= xy - bz \end{aligned}$$

For $a = 10, b = 8/3, c = 28$, Lorentz observed a **strange attractor**.

THE ROESSLER SYSTEM. The following system of differential equations in space was found by Otto Rössler in 1976. The system was designed as a model for a strange attractor without any application in mind. It is theoretically interesting because a return map resembles the one dimensional logistic map $f_c(x) = cx(1 - x)$:

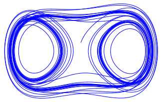


$$\begin{aligned} \dot{x} &= -(y + z) \\ \dot{y} &= x + 0.2y \\ \dot{z} &= 0.2 + xz - cz \end{aligned}$$

It is parametrized by a parameter c. The picture to the right shows an orbit for $c = 5.7$. For parameters in the range $2.5 < c < 10$ one observes a Feigenbaum bifurcation scenario.

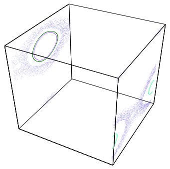
THE DUFFING SYSTEM $\ddot{x} + b\frac{\dot{x}}{}x + x^3 - c\cos(t) = 0$ can be written as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -by - x + x^3 - c\cos(z) \\ \dot{z} &= 1 \end{aligned}$$



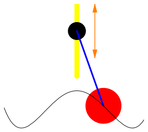
The Duffing system models a metallic plate between magnets. It is a harmonic oscillator with an additional cubic force, some damping and an external periodic driving force.

THE ABC FLOW. It is a flow with three parameters a, b, c, therefore its name **ABC flow**. An other etymological explanation is that Arnold, Beltrami and Childess worked on this system. Even so the system looks simple, its solutions can be complicated.



$$\begin{aligned} \dot{x} &= a\sin(z) + c\cos(y) \\ \dot{y} &= b\sin(x) + a\cos(z) \\ \dot{z} &= c\sin(y) + b\cos(x) \end{aligned}$$

FORCED PENDULUM. The differential equation $\ddot{x} = \cos(x) + g\sin(t)$ describes a pendulum which is periodically shaken up and down. The equations



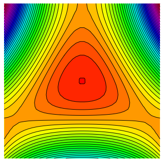
$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \cos(x) + g\sin(z) \\ \dot{z} &= 1\ . \end{aligned}$$

in space have a natural Poincare section $z = 0$.

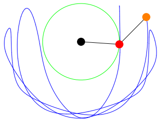
HENON-HEILS SYSTEM. The differential $\frac{dx_i}{dt} = H_{y_i}(x, y)$, $\frac{dy_i}{dt} = -H_{x_i}(x, y)$ with

$$H(x, y) = \frac{1}{2}(y_1^2 + y_2^2 + x_1^2 + x_2^2) + x_1^2 x_2 - \frac{1}{3}x_2^2$$

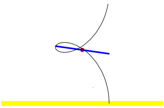
was studied first numerically by Henon and Heils in 1964. Energy surfaces $\{H(x, y) = E\ \}$ are invariant. For $0 \leq E \leq 1/6$ the surface is bounded and solutions stay bounded. The Poincaré section $\Sigma = \{x_1 = 0\}$ defines an area-preserving map on a subset of the plane.



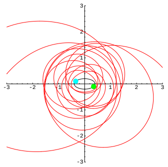
DOUBLE PENDULUM. The double pendulum is described by four variables. Energy conservation defines a differential equation on a three dimensional space. The return map $x = 0$ defines a map on the cylinder. If the gravitational field is zero, the double pendulum is integrable. With gravity $g > 0$, the system is complicated.



FALLING COIN. A falling coin defines a dynamical system which is often used, to produce random events: you flip a coin or dice and let it hit the ground, where it bounces. Flipping a coin and catching by the hand uses an integrable system. Some people can throw, catch and predict the outcome. If the stick moves in a gravitational field and if there are no impacts, then there is besides energy conservation also momentum conservation: the system becomes integrable. With impact, the system develops chaos.



3 BODY PROBLEM. The restricted three body problem in the plane is the situation, where the third particle is assumed not to influence the two other bodies. By Kepler, the two bodies moves on ellipses and produce a time periodic force on the third body. Therefore, we obtain a differential equation of the form $\frac{d^2}{dt^2}\vec{x}(t) = F(\vec{x}, t)$, where $\vec{x} = (x, y, \dot{x}, \dot{y})$. Energy conservation defines a three dimensional system.



STOERMER PROBLEM. A charged particle in a magnetic dipole field has rotational symmetry and so an angular momentum integral. This allows to reduce the system to a differential equation with four variables. The energy integral defines a flow on a three dimensional space. The system can be studied using a return map. The relevance of the system is the motion of charged particles in the **van Allen belts** and the explanation of the **Aurora Borealis**.