12.4 The Lorenz Attractor

The Rössler system is an artificial system designed solely with the purpose of creating a model for a strange attractor which uses only the simplest chaos generating mechanism, stretch-and-fold. Of course, Rössler knew about the Lorenz system, which had been published 13 years before. In fact, we may say the Rössler attractor is a model of the Lorenz model.

Edward N. Lorenz

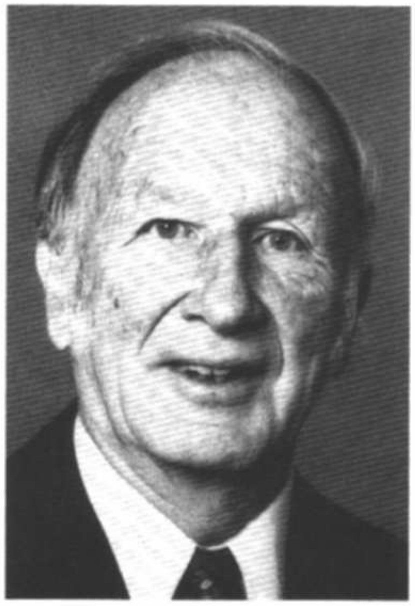


Figure 12.32

The Lorenz System

The system of equations that Lorenz proposed does not look any more complicated than that of Rössler. Here it is:

$$\begin{aligned} x' &= -\sigma x + \sigma y \\ y' &= Rx - y - xz \\ z' &= -Bz + xy \end{aligned} \qquad (12.13)$$

The numbers σ, B, and R are the system's physical parameters, which Lorenz fixed at

$$\sigma = 10, \quad B = \frac{8}{3}, \quad R = 28 \; .$$

Figure 12.33 shows the corresponding attractor, which is now called the Lorenz attractor. Clearly the geometry is more involved than for the Rössler band. There are two sheets in which trajectories spiral outwards. When the distance from the center of such a spiral becomes larger than some particular threshold, the solution is ejected from the spiral and attracted by the other spiral, where it again begins to spiral out and the game is repeated. The number of turns that a trajectory spends in one spiral and then in the other is not specified. It may wind around one spiral two times, then three times

The Lorenz Attractor

Some trajectories from the Lorenz attractor.

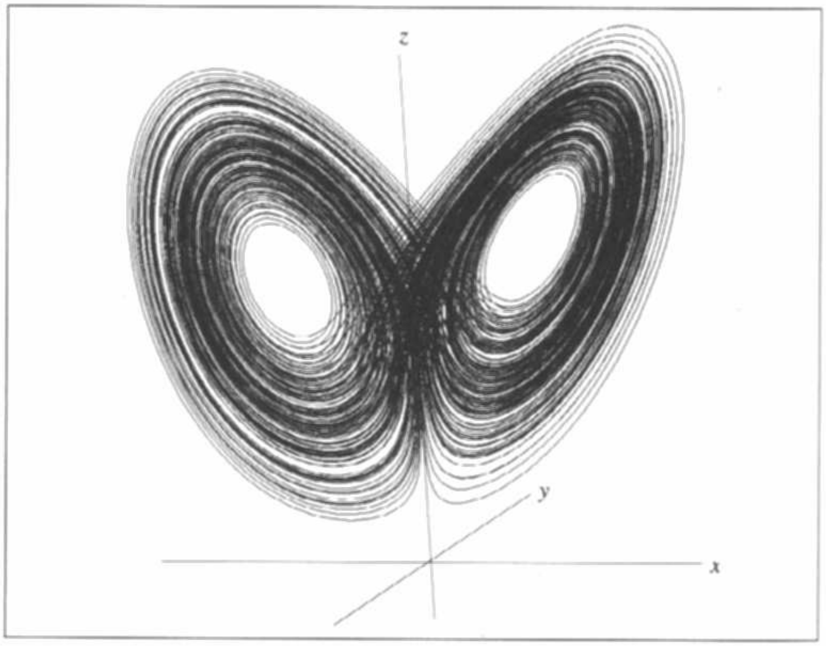


Figure 12.33

around the other, then ten times around the first and so on. In fact, we believe that for any sequence of positive numbers which are not too large, for example 3, 11, 7, ... there exists a trajectory on the Lorenz attractor with precisely these numbers as turns around the spirals. Thus, there is a solution that turns 3 times around the right spiral, then 11 times around the left, then 7 times around the right again and so on.

Lorenz's Physical Model

What is the connection between these wildly spinning solutions and weather forecasting which is what Lorenz was interested in? Certainly, the trajectories should not be mistaken for the paths of air currents! If this were the case then the Lorenz attractor would act similar to a black hole in astrophysics, sucking in all the atmosphere — leaving nothing but emptiness around it and laying waste to the whole planet Earth. But we are not far from the truth. The Lorenz system is in fact a model of thermal convection which, however, includes not only a description of the motion of some viscous fluid or atmosphere but also the information about distribution of heat, the driving force of thermal convection.

When air is warmed near the Earth's surface it rises. This is an important factor in the atmospheric weather factory. Convection air currents may accumulate and give rise to convection cells of several types and, when forced more vigorously, may produce very turbulent motion in the atmosphere. Examples of convection cells are cylindrical rolls and structures, which are called *Bénard cells*, resembling a honeycomb from above. In these hexagonal cells the warmed portions of the fluid rise in the center, get colder near the top and sink back down to the surface around the boundary of the cell. The Lorenz system is related more to the cylindrical roll type of fluid motion in which one of the dimensions can be disregarded pretending that these rolls extend to infinity. The mathematical model of the fluid motion had originally been developed by Lord Rayleigh[30] in 1916. It assumes that all convection happens in a rectangular region whose bottom is heated such that the temperature difference between bottom and top remains constant. With certain parameter configurations in this model it turns out that the solutions to the model equations have a rather special form which was already known to Rayleigh. Lorenz took these special solutions, regarded their amplitudes as time-dependent, inserted them in the Rayleigh model, disregarded all terms that are not in this special form,[31] and arrived at a system of differential equations for the time-dependent amplitudes, eqn. (12.13).[32]

The Meaning of x, y and z

It is almost impossible to find out what the variables x, y, and z in the Lorenz system precisely stand for without consulting Lorenz' original paper. Let us give some technical details and visualizations of the convection process which was investigated by Lorenz. As already stated we deal with convection currents and temperature distribution in a rectangular region where the temperature difference ΔT between the bottom and the top is kept constant. The dynamics are assumed to be identical in all slices parallel to the rectangular region. The governing equations for the more general three-dimensional problem were worked out by Lord Rayleigh. The simplification to the two-dimensional case considered here is by B. Saltzman.[33]

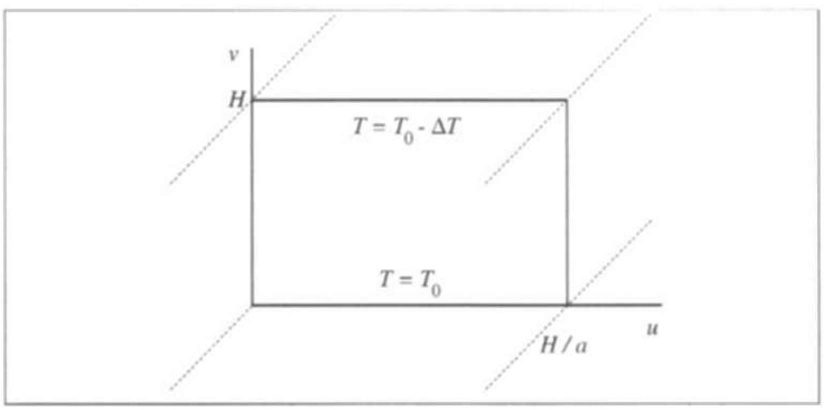


Figure 12.34 : The coordinate system in the cross-section of a bar where the Lorenz equations present a model for fluid flow and temperature.

[30]Lord Rayleigh, *On convective currents in a horizontal layer of fluid when the higher temperature is on the under side*, Phil. Mag. 32 (1916) 529–546.

[31]Except for one particular term related to the temperature distribution.

[32]Another interpretation is that the solutions of the Rayleigh equations can be written as Fourier series with time-dependent coefficients. Using these series in place of the original variables produces a system containing infinitely many equations. When keeping only the three most significant of these we again obtain the Lorenz system given in eqn. (12.13).

[33]B. Saltzman, *Finite amplitude free convection as an initial value problem — I*, J. atmos. Sci. 19 (1962) 329–341.

We do not include these formulas but instead, present the type of approximation of a solution used by Lorenz. It involves a so-called stream function Ψ and a temperature function. The variables in these functions are the spatial coordinates u and v, and time t. In place of the actual temperature, the difference Θ with a temperature profile belonging to the state of no convection is used, i.e., where the temperature decreases linearly from some value T_0 at the bottom to $T_0 - \Delta T$ at the top (see figure 12.34).

Let us give these complicated looking equations and then explain.

$$\frac{a}{(1+a^2)\kappa}\Psi(u,v,t) = x(t)\sqrt{2}\sin\left(\tfrac{\pi a}{H}u\right)\sin\left(\tfrac{\pi}{H}v\right)$$

$$\frac{\pi R_a}{R_c\Delta T}\Theta(u,v,t) = y(t)\sqrt{2}\cos\left(\tfrac{\pi a}{H}u\right)\sin\left(\tfrac{\pi}{H}v\right) - z(t)\sin\left(\tfrac{2\pi}{H}v\right) \qquad (12.14)$$

The symbols used have the following interpretation.

$\Psi = \Psi(u,v,t)$	stream function
$\Theta = \Theta(u,v,t)$	local temperature difference
u	horizontal spatial coordinate
v	vertical spatial coordinate
$x(t), y(t), z(t)$	time-dependent coefficients (amplitudes)
t	time
H	depth of fluid layer (maximum of v)
a	parameter of geometry (fixed at $a = 1/\sqrt{2}$)
κ	thermal conductivity
R_a	Rayleigh number
R_c	critical value of R_a ($R_c = \pi^4(1+a^2)^3/a^2$)
ΔT	total temperature difference

The rectangular region has coordinates u ranging from 0 to $H/a = \sqrt{2}H$ and v ranging from 0 to H. The stream function Ψ is interpreted in the following sense. Ψ is a scalar field and the fluid motion at a time t_0 occurs along the isolines $\Psi(u,v,t_0) = const.$ Thus, to obtain a picture of the fluid motion we can simply plot these isolines (see figure 12.35). More precisely, the velocity V of the fluid at a given point (u,v) in space and t in time is

$$V(u,v,t) = \begin{pmatrix} -\dfrac{d}{dv}\Psi(u,v,t) \\ \dfrac{d}{du}\Psi(u,v,t) \end{pmatrix}$$

The corresponding temperature profile is given by $\Theta(u,v,t)$ and can be read directly from the formula (12.14).

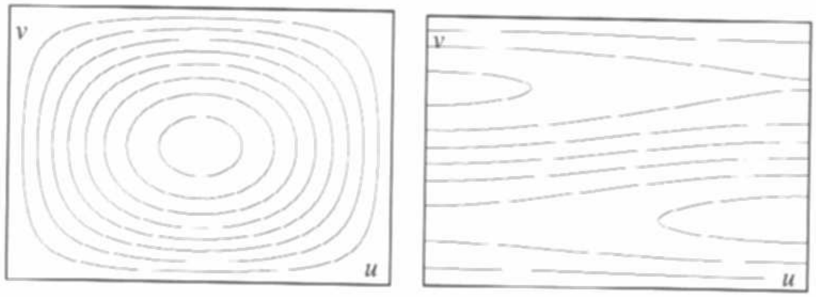


Figure 12.35 : Streamlines of convection currents (left) and corresponding temperature profile (right) at steady state.

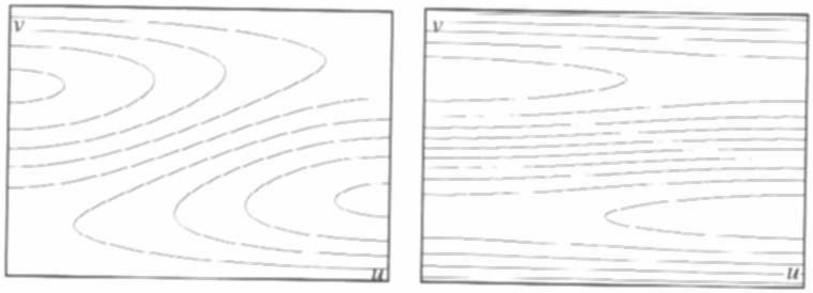


Figure 12.36 : The corresponding state variables (x, y, z) are $(2.403, 4.892, 4.673)$ (left) and $(16.610, 7.428, 45.428)$ (right).

Using this form of solution, Saltzman's equations reduce to the system of Lorenz which contains only the unknown time-dependent coefficients. Recall

$$\begin{aligned} x' &= -\sigma x + \sigma y \\ y' &= Rx - y - xz \\ z' &= -Bz + xy \end{aligned}$$

which includes three parameters: σ, the Prandtl number, $R = R_a/R_c$ and $B = 4/(1 + a^2) = 8/3$.[34] Lorenz explains: 'In these equations x is proportional to the intensity of the convective motion, while y is proportional to the temperature difference between the ascending and the descending currents, similar signs of x and y denoting that warm fluid is rising and cold fluid is descending.'

Lorenz chose the parameters $\sigma = 10$ and $R = 28$. For this setting there are two steady states of the differential equation, i.e., values of x, y, z which remain constant (besides the origin $(0, 0, 0)$). These are

$$(6\sqrt{2}, 6\sqrt{2}, 27) \text{ and } (-6\sqrt{2}, -6\sqrt{2}, 27)$$

and correspond to the centers of the two 'holes' in the attractor shown in figure 12.33. Associated with these solutions are steady

[34]The derivatives x', y', z' are with respect to reparametrized time τ, namely $\tau = \pi^2(1 + a^2)\kappa t/H^2$.

states of convection. We illustrate the first of these in figure 12.35. Other points in phase space correspond to other convection currents and temperature profiles. Two more examples for points on the Lorenz attractor are given in figure 12.36. Thus, when we follow a point along its trajectory on the Lorenz attractor, we must interpret its coordinates in this sense as given by the formulas in eqn. (12.14) and the figures. With these remarks we conclude the discussion of the modeling aspect of the Lorenz system.[35]

Model of the Dynamics with Lorenz Map

A schematic diagram of the stretch-split-and-merge process in the dynamics on the Lorenz attractor is shown at the top. The Lorenz map of the system shown below models the stretch-split-and-merge process as observed at the interval I. Each (graphical) iteration corresponds to a turn around one of the lobes of the attractor.

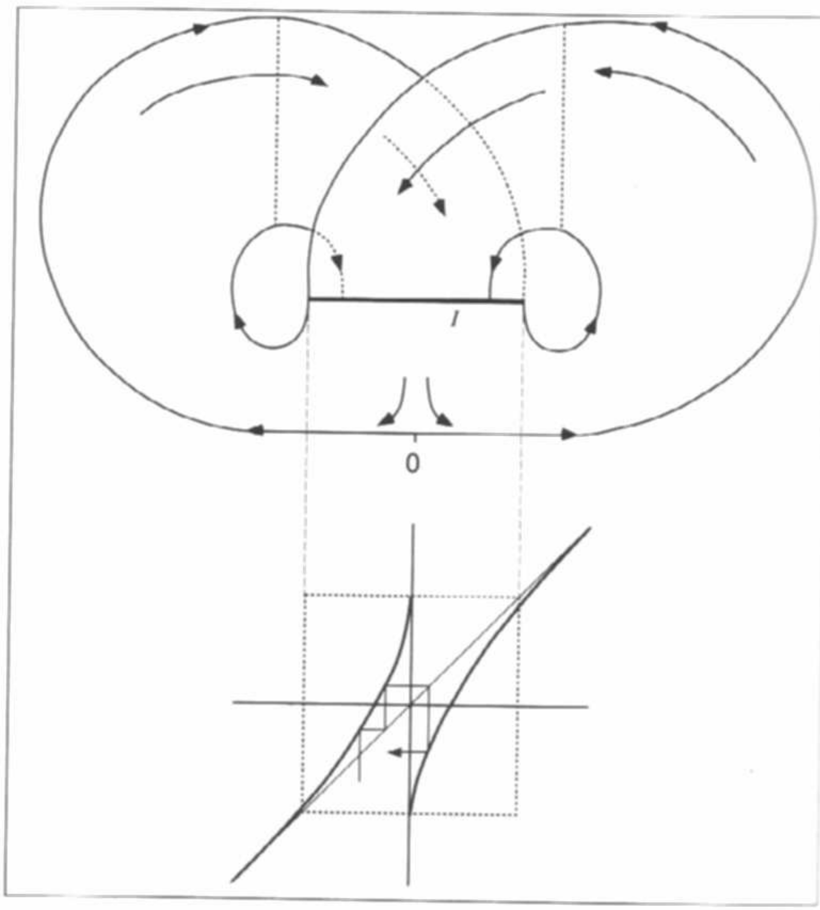
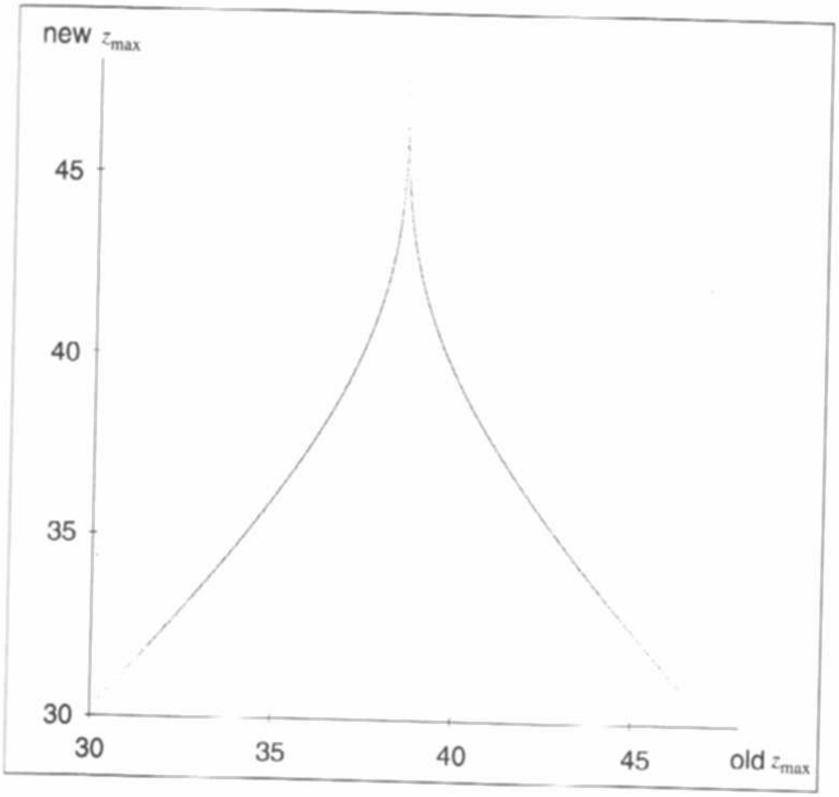


Figure 12.37

As outlined, there are several severe simplification steps before we get to the final set of equations, and we may rightfully say that the solutions of the system may not bear any significance for the real convection process. But it was not Lorenz' intention to be as precise as possible in the modeling. On the contrary, after having discovered a strange attractor (in a more complex system) he strived for the most elementary system that can be derived

[35]For more details and references see the original paper of E. N. Lorenz, *Deterministic nonperiodic flow,* J. Atmos. Sci. 20 (1963) 130–141. The book by J. M. T. Thompson and H. B. Stewart, *Nonlinear Dynamics and Chaos,* Wiley, Chichester, 1986, contains a broad introduction and much more material about the geometry underlying the attractor and its route to chaos.



Lorenz Map

This Lorenz map for the Lorenz system models the dynamics of the attractor as observed at the dotted vertical lines pictured in figure 12.37. From this point of view it can be described by a stretch-and-fold process.

Figure 12.38

from the convection equations and that would still demonstrate the extreme sensitivity to initial conditions, which since has become the trademark of chaos.

A Model for the Lorenz Dynamics

The chaos generating mechanism in the Lorenz system is a bit more involved than the one in the Rössler system. Rather than featuring a stretch-and-fold action, we have a stretch-split-and-merge operation as shown in the model in figure 12.37. Around the two spirals a stretching takes place. Both stretched bands split near the horizontal line in the center, one half of them returning to the left spiral, the other to the right. During the subsequent turn, the two bands on each part of the attractor merge, and the cycle is completed. Similar to the Lorenz map for the folded band, a Lorenz map for points on the central line segment can now be defined. The result is shown in the lower part of the figure.

The graphical iteration using this graph corresponds to the dynamics of points (x, y, z) on the attractor. The left half of it belongs to the left spiral of the attractor, the other to the right. There is a connection between this one-dimensional model and the shift transformation in chapter 10, $x \to \mathrm{Frac}(2x) = 2x \bmod 1$, from which we deduced the essential properties of chaos: sensitive dependence on initial conditions, dense periodic points, and mixing.

A Periodic Solution

A stable, periodic solution to Lorenz' system at parameter $R = 100.5$, projected onto the xz-plane.

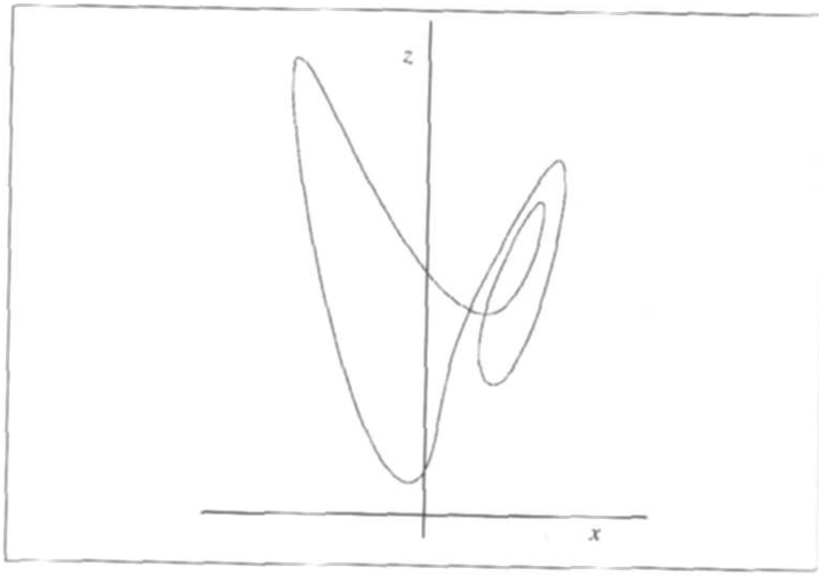


Figure 12.39

Another Lorenz Map

Originally Lorenz had proposed a different one-dimensional model. He observed that a trajectory "leaves one spiral only after exceeding some critical distance from its center. Moreover, the extent to which this distance is exceeded appears to determine the point at which the next spiral is entered; this in turn seems to determine the number of circuits to be executed before changing spirals again." He then concluded that the maximum z-value alone suffices to predict the maximum for the following circuit. To check this idea, he plotted many points with coordinates being two consecutive maximum z-values of a trajectory as in figure 12.38; thus was born the first Lorenz map.[36] The points appear to lie on a curve and graphical iteration can be used to predict the maximum z-value of the following spiral turns if the current one is given. The similarity of the graph to the tent function is quite apparent and Lorenz' paper continues with a short study of the chaotic dynamics associated with the tent function and concludes that there must be an infinity of trajectories corresponding to what he called nonperiodic deterministic flow, i.e., that one now calls chaos.

Cantor Set Structure and Fractal Dimension

Again, it is true that the Lorenz maps from figures 12.37 and 12.38 are only models for a truly two-dimensional Poincaré map. If such one-dimensional models were exact, this would imply a perfect merging of the two 'spiral surfaces' which contradicts the uniqueness of solutions of Lorenz' system of differential equations. Similar to the bands in the Rössler attractor, these surfaces come very, very close to each other, indistinguishable to the eye, but they cannot completely merge. Again, there is a Cantor

[36]There is a surface which contains all points (x, y, z) with a maximum z-value of a corresponding trajectory. At such a maximum the derivative z' must necessarily vanish. Thus, from eqn. (12.13), $xy - Bz = 0$, or $z = xy/B$. This equation describes a surface, the graph of the function xy/B. A portion of this surface may be interpreted as a proper Poincaré section of the Lorenz system.

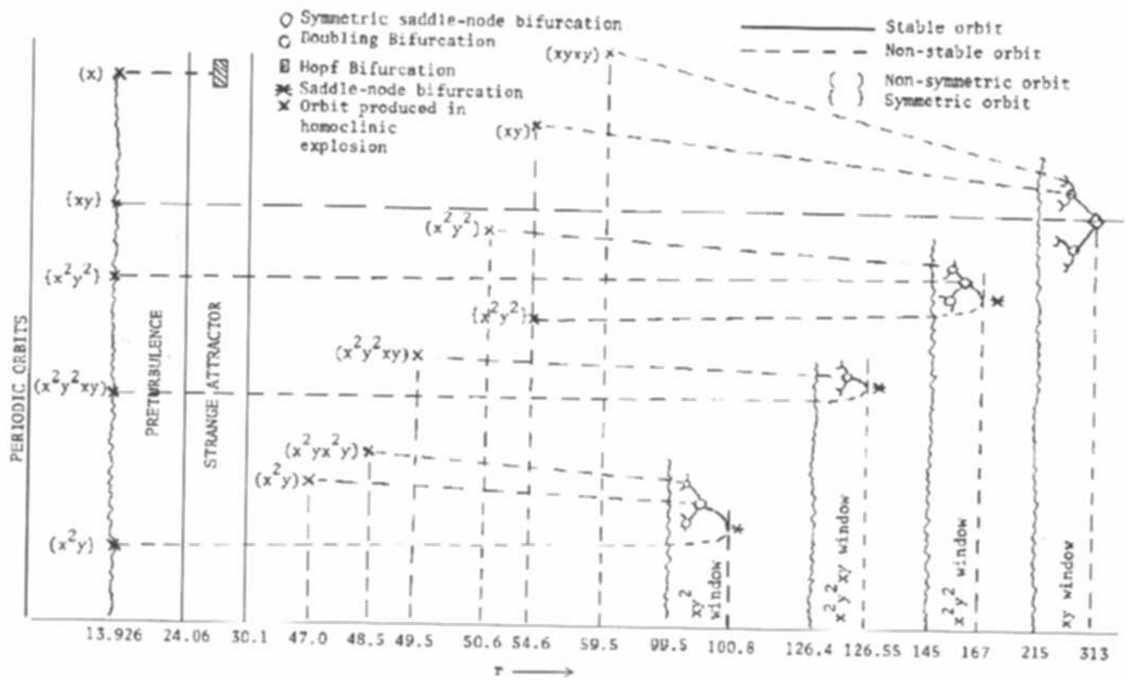


Figure 12.40 : Bifurcation diagram showing a few of the shorter periodic trajectories. Reproduced with permission of the publisher from C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*, Springer-Verlag, New York, 1982, page 99.

set structure of these surfaces leading to a fractal dimension only slightly above 2.[37]

Changing the Parameter R

Let us conclude this section by looking at phenomena brought to light by changing the parameter R in the Lorenz system. There are a large number of periodic solutions, some of which are stable and attracting, while others are unstable and repelling. For some parameters, chaos and stable equilibria coexist. The whole palette of features worth discussing cannot be included here; it would take up too much space. Here we can only highlight a couple of aspects; for further details we can only point to the literature.[38]

Period Doubling Window and Feigenbaum Number

We begin by looking at the range of parameters $99.524 < R < 100.795$. Here we can observe a period-doubling scenario similar to the one in the Rössler attractor. This cascade of bifurcations occurs as we decrease the parameter R. Figure 12.39 shows an attractive periodic orbit obtained for $R = 100.5$. It spirals around twice in the 'positive' half space $x > 0$, then one time in the negative half space $x < 0$ before it repeats. This solution would be called an x^2y-solution following the naming conventions in some of the literature.[39] When we lower the parameter below $R \approx 99.98$

[37]The dimension has been estimated as 2.073. See E. N. Lorenz, *The local structure of a chaotic attractor in four dimensions*, Physica 13D (1984) 90–104.

[38]For example, see the book by C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*, Springer-Verlag, New York, 1982.

[39]In the code $x^k y^l$ an x stands for a turn around the fixed point in the half space $x > 0$, while the symbol y denotes a turn

Intermittency

Intermittency in the Lorenz system. For the parameter $R = 166$ there is a periodic solution (top), while for $R = 166.2$ solutions appear similar, however, interrupted by sudden chaotic bursts (bottom).

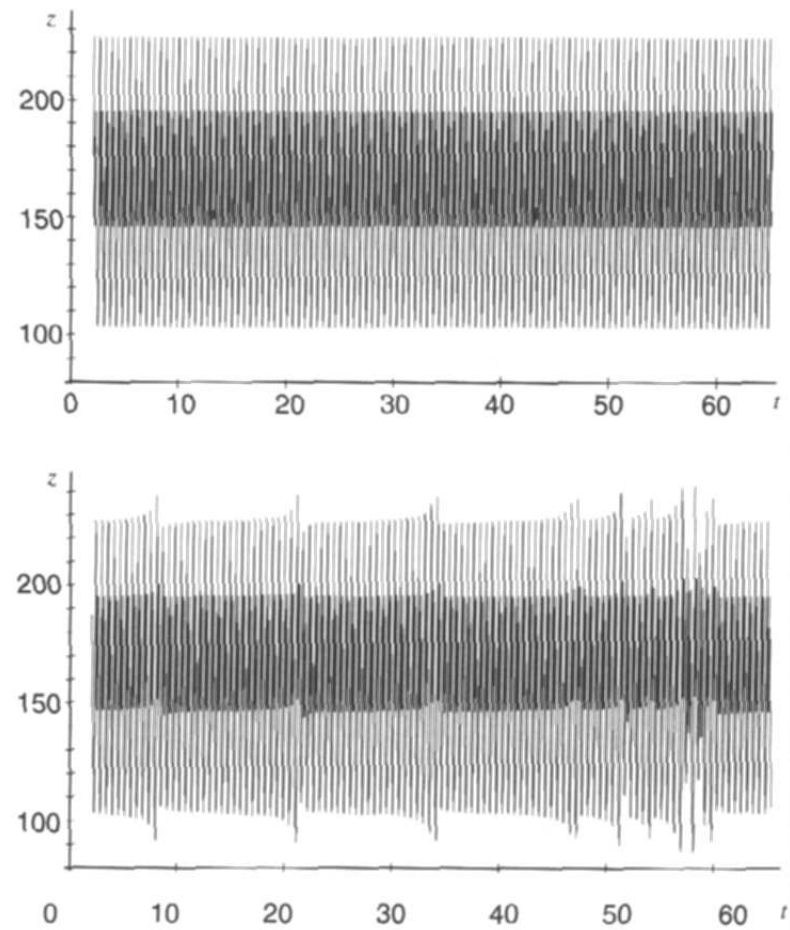
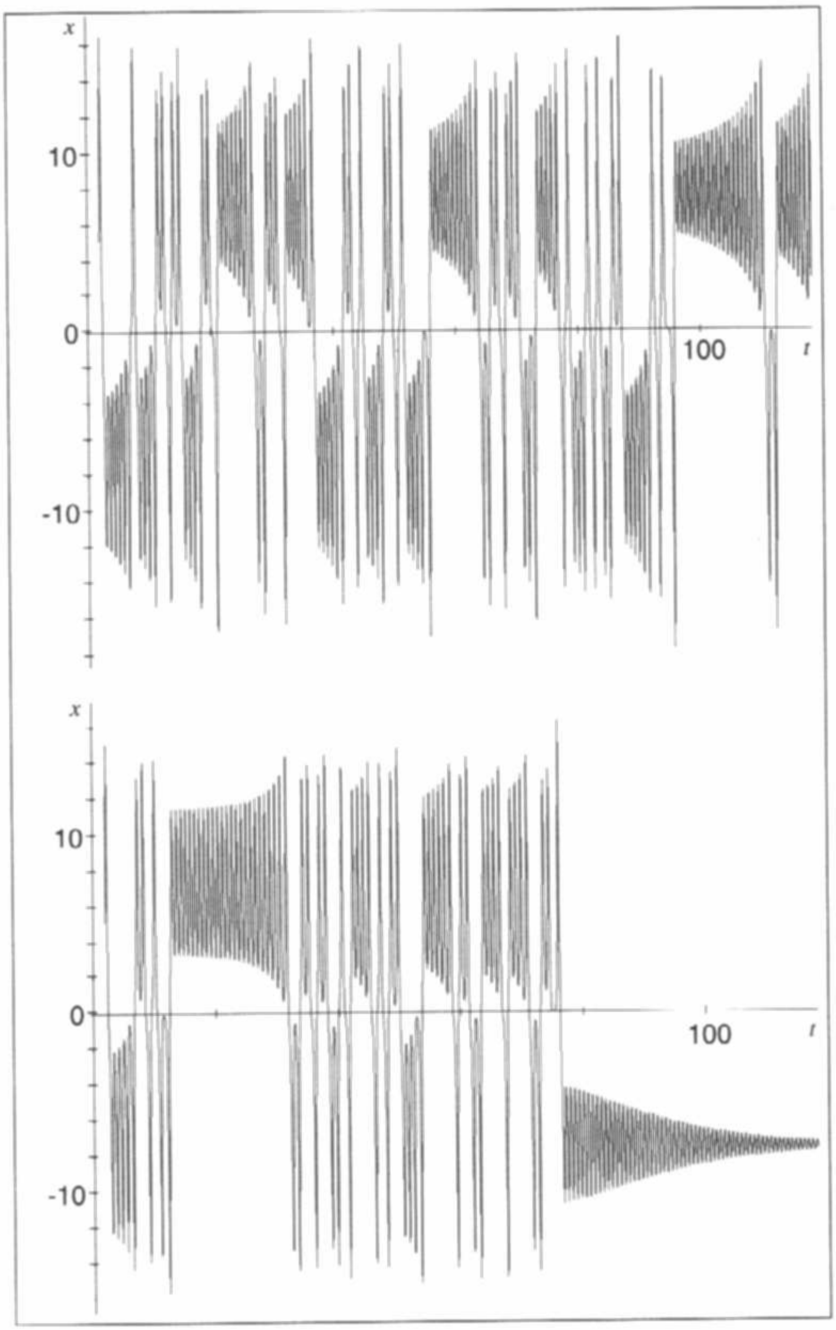


Figure 12.41

this solution doubles up to a periodic solution of twice the period and type $x^2yx^2y = (x^2y)^2$. This happens in the same spirit as in the bifurcation of periodic solutions in Rössler's system. Further bifurcations of period-doubling can be observed when we continue to lower the parameter, as listed in the following table.

Parameter R	Type of Solution
100.795	x^2y
99.98	$(x^2y)^2$
99.629	$(x^2y)^4$
99.547	$(x^2y)^8$
99.529	$(x^2y)^{16}$
99.5255	$(x^2y)^{32}$

in the other half space. Thus, in this example there are k turns of type x followed by l turns of type y. The notation can be extended to longer symbol strings such as xy^2xy^3 and so on.



A Crisis of the Lorenz Attractor

For $R = 25$ (top) there is a strange attractor for the Lorenz system. When the parameter is lowered to $R = 22.4$ (bottom) the attractor undergoes a crises. Trajectories appear chaotic only in an initial transient phase. After that one of the attractive rest points is approached.

Figure 12.42

Based on more precise calculations the Feigenbaum number δ belonging to this sequence has been estimated.[40] The result is $\delta \approx 4.67$, a number which at this precision is indistinguishable from Feigenbaum's famous ratio 4.6692... There are other windows in the parameters R with period-doubling bifurcations. To give an impression of the complicated scenarios,

[40] V. Franceschini, *A Feigenbaum sequence of bifurcations in the Lorenz model*, Jour. Stat. Phys. 22 (1980) 397–406.

we reproduce a chart (figure 12.40) compiled by Colin Sparrow.

Intermittency

Knowing that the dynamics of Lorenz' and Rössler's systems have a strong connection to the one-dimensional iteration of quadratic transformations, it is no longer a surprise to again find the back doors to chaos that we discussed in chapter 11: intermittency and crises. Recall that intermittency means that a solution spends most of its time near a periodic solution but is interrupted by sudden and erratic chaotic bursts. For an only slightly perturbed parameter value, these bursts disappear and only the periodic behavior remains. Exactly the same can be observed, for example, in the Lorenz system for the parameters $R = 166.2$ and $R = 166.0$ (see figure 12.41).

Crises

When a periodic solution 'disturbs' the chaos, an intermittent trajectory is produced. In this case, the chaos prevails. However, the chaotic attractor deteriorates, when a periodic trajectory steals the attractivity from the attractor. What remains of the chaos, is the long transient chaotic behavior of solutions, all of which approach the periodic one asymptotically. The chaotic attractor is said to be in a crisis. And again this can also be discovered in the Lorenz system (see figure 12.42). These phenomena are by no means restricted to the quadratic iterator and the Lorenz system; they have been identified in many other mathematical and physical dynamical systems.[41]

[41] See T. Tél, *Transient chaos, Directions in Chaos III*, Hao B.-L. (ed.), World Scientific Publishing Company, Singapore.