

# Cellular Automata Offer New Outlook on Life, the Universe, and Everything

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What kind of world do we live in? The question has been bandied about for thousands of years by philosophers, theologians, and politicians. More recently, a spectrum of talk show hosts have weighed in on the subject. So far, no one's come up with an answer that everyone can agree on.

Mathematicians have considered the same question. But where others worry over the blurred boundaries of Good and Evil, mathematicians ponder a sharper dichotomy: the Continuous versus the Discrete.

Continuous mathematics, exemplified by calculus and differential equations, has long dominated mathematical descriptions of the world. But discrete mathematics is making a bid for primacy. With modern computers, researchers have discovered astonishingly complex behavior in seemingly simple, finite systems. The results have led some theorists to speculate that discrete models, which lend themselves to digital computation, are the "right" way to study nature.

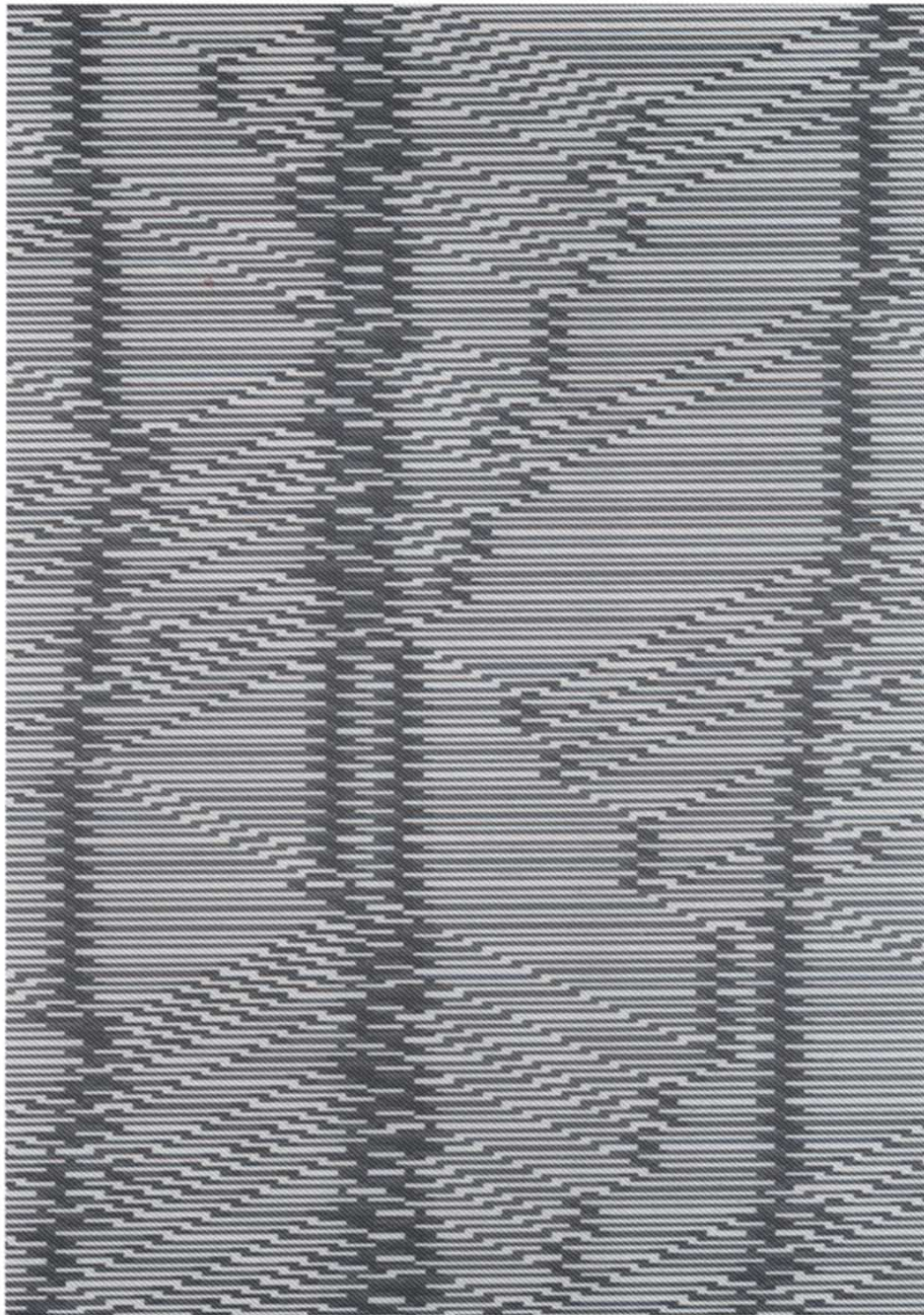
Erica Jen, a mathematician at Los Alamos National Laboratory in Los Alamos, New Mexico, is one of a growing number of researchers who believe that discrete mathematics can mirror many aspects of physical reality fully as well as the more customary continuous theories. Jen has been studying mathematical properties of discrete systems known as cellular automata. These systems, she says, are useful models for many types of complex physical, chemical, or biological systems. They also have an amazing life of their own.

Cellular automata "exhibit an extremely rich and diverse range of pattern formation," Jen says. Among the most interesting are "self-organizing" patterns: highly structured features that seem to emerge spontaneously from a "primordial soup" of random binary

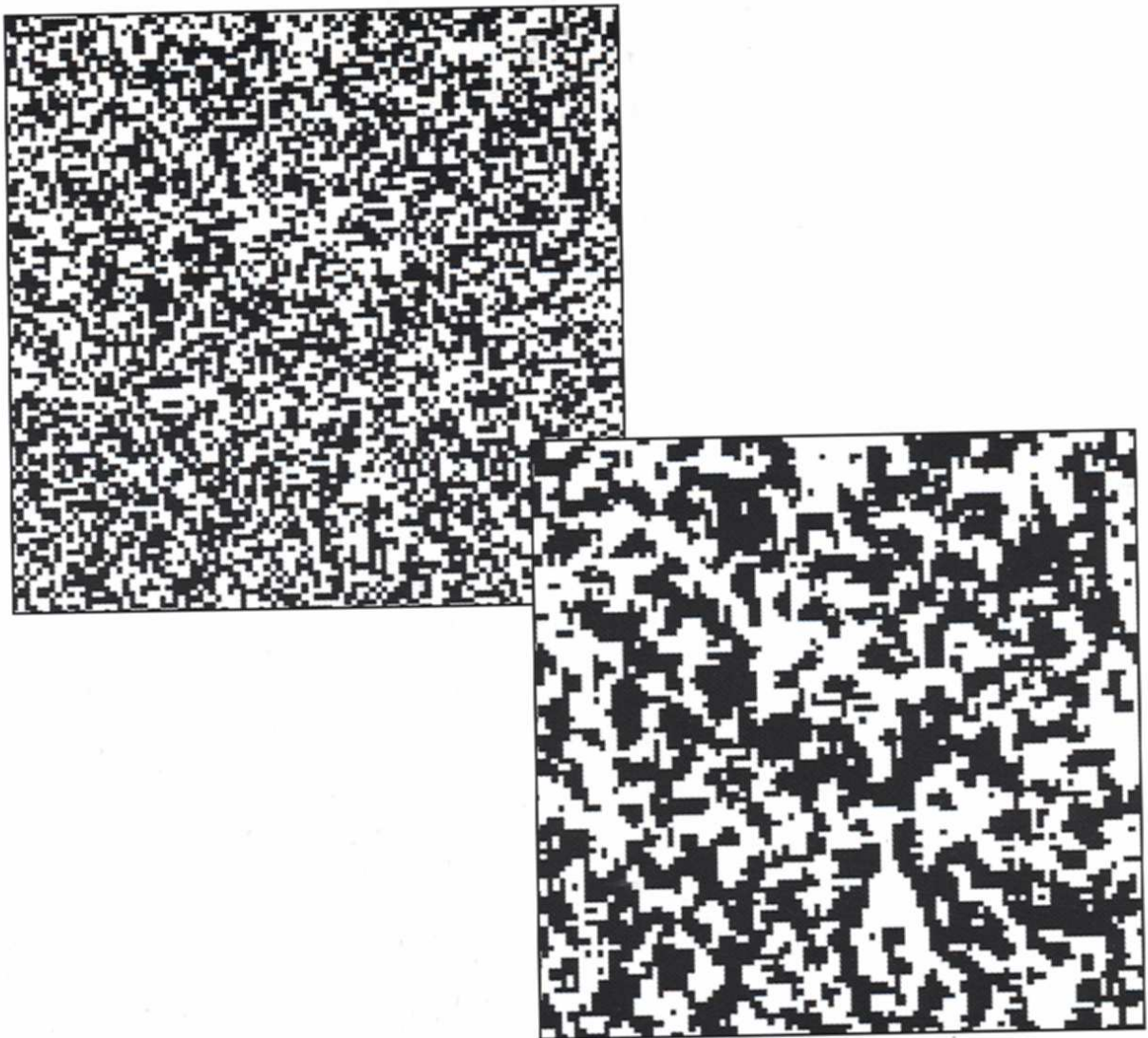


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Erica Jen. (Photo courtesy of Erica Jen.)



**Figure 1.** Domain walls and criss-crossing "particles" on a richly textured background are features of one-dimensional cellular automata based on Rule 54 (see page 78). (Figure courtesy of Erica Jen, created using SigniScope™ software package, courtesy of Signition, Inc.)



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**Figure 2.** A  $100 \times 100$  “majority vote” cellular automaton proceeds from a random initial state (top) to a final state (bottom right, facing page). On each ballot, every cell looks at the cells around it, and changes color value if its current value is in the minority. Most cells have 8 neighbors, but cells on the edges have 5 neighbors, and corner cells only 3. Some features of the final state take shape with the first round of voting (middle).

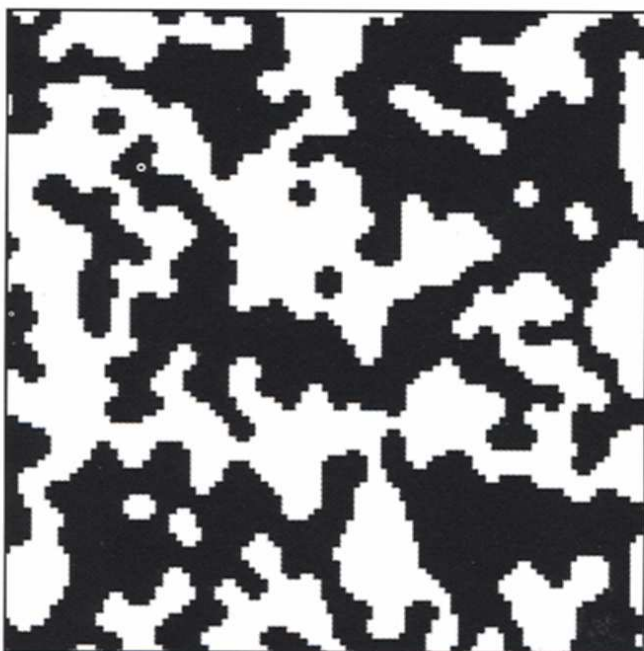
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digits. Jen and her colleagues hope to understand exactly how these patterns arise and precisely what properties they possess. By studying cellular automata with mathematical tools from areas such as abstract algebra and number theory, Jen hopes to bring theoretical rigor to a subject that is often as much art as science.

Loosely speaking, a cellular automaton is a “pixelization” of space and time.<sup>8</sup> Instead of varying continuously from point to point and moment to moment, cellular automata consist of discrete “cells” with discrete values that change instantaneously at discrete intervals, much like frames in a movie. The crucial feature, moreover, is a *rule* that prescribes exactly how each cell’s value changes depending on the values of nearby cells.

One possible rule, for example, is a “majority vote”: Each cell in a system of black and white squares could be programmed to switch color if the majority of its immediate neighbors are of the opposite color (see Figure 2). Another rule might specify that the value of each cell change to the sum of the values of the cells surrounding it—or, reducing things to black and white again, to the *parity* of the sum (black could be odd and white even).

“The essential features of cellular automata are that they are deterministic and discrete in space, time, and state values; they evolve according to local interaction rules; and these rules apply



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synchronously and homogeneously across the system," Jen explains. These features accord well with standard physical assumptions about the uniformity of space and time (the laws of physics are the same everywhere) and the impossibility of instantaneous action at a distance (nothing travels faster than the speed of light). They also lend themselves to modeling complex systems consisting of a large number of simple components that are locally connected. Perhaps most important, these features are tailor-made for digital computation.

Cellular automata were first dreamt of in the early 1950s by John von Neumann and Stanislaw Ulam, as tools for studying biological systems. In the late 1960s, John Conway, then at Cambridge University (now at Princeton), invented rules for a cellular automaton he called the Game of Life, which Martin Gardner popularized in his column for *Scientific American*. But computer technology was not really up to the job of exploring cellular automata until the 1980s, when color graphics workstations replaced the clattering teletype machines that traded alphanumeric symbols with a room-sized mainframe in another building.

With today's high-speed machines (fated, no doubt, to seem painfully slow in another few years), researchers can glimpse the complex patterns that often arise from the repeated application of the simple rules that define cellular automata. Fast computers allow experiments with relatively large systems: Automata with thousands of cells can be followed for hundreds of time steps on a personal computer; workstations and supercomputers can track systems with millions of cells for thousands of time steps.

Jen's research focuses on a class of one-dimensional systems called "elementary" cellular automata. Each state of such a system is represented by a row of black and white pixels, corresponding to a string of 1's and 0's, and the update rule uses only the value of a given cell and the values of its two adjoining cells. (To simplify the description, researchers often work with a "wrap-around" model, in which the two ends are joined, so that all cells are treated alike.) The evolution of a one-dimensional automaton is conveniently displayed in a two-dimensional format, each new row below its predecessor. (Researchers also often "colorize" their elementary systems to highlight key features.) The result can be as richly textured as a Navajo weaving.

In the early 1980s, Stephen Wolfram, then at the Institute for Advanced Study in Princeton, roughed out a classification scheme

for elementary automata based on the two-dimensional patterns that emerge from “random” initial states. Some rules, he noted, lead quickly to uninteresting, static behavior. The majority vote rule, for example, simply chips away at any segments of alternating values, and stops changing once it removes them. But other rules, Wolfram found, produce patterns that seem to reflect elements of both order and chaos.

Wolfram also introduced a convenient notation for the rules of one-dimensional elementary cellular automata. Because every rule amounts to assigning a 1 or a 0 to each of the eight strings 111, 110, 101, 100, 011, 010, 001, and 000, each rule can be identified with an 8-bit binary number (see Figure 3). For example, the majority vote rule corresponds to the binary number 11101000, or 232 (= 128+64+32+8). In particular, there are only 256 possible rules for one-dimensional elementary automata. (Wolfram cut the number to 32 by restricting attention to those rules that assign 0 to 000, thus leaving “empty space” empty, and respect left–right symmetry by assigning the same values to 100 and 001 and to 110 and 011.) The open-endedness of the subject comes from the fact that there can be arbitrarily many (or even infinitely many) cells.

Computer experiments are commonly used to investigate the kinds of patterns that are associated with the 256 rules. Surprisingly, researchers have found that rules generate certain “generic” types of behavior: Except for specially engineered initial conditions (for example, an initial state consisting of strictly alternating 1’s and 0’s, or, worse, all 0’s), each rule will typically

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	111	110	101	100	011	010	001	000
18	0	0	0	1	0	0	1	0
54	0	0	1	1	0	1	1	0
90	0	1	0	1	1	0	1	0
126	0	1	1	1	1	1	1	0
146	1	0	0	1	0	0	1	0

**Figure 3.** Each rule for an elementary one-dimensional cellular automaton can be read as a base-2 expression for a number between 0 and 255. The leftmost column gives the common, base-10 name for five especially interesting rules.

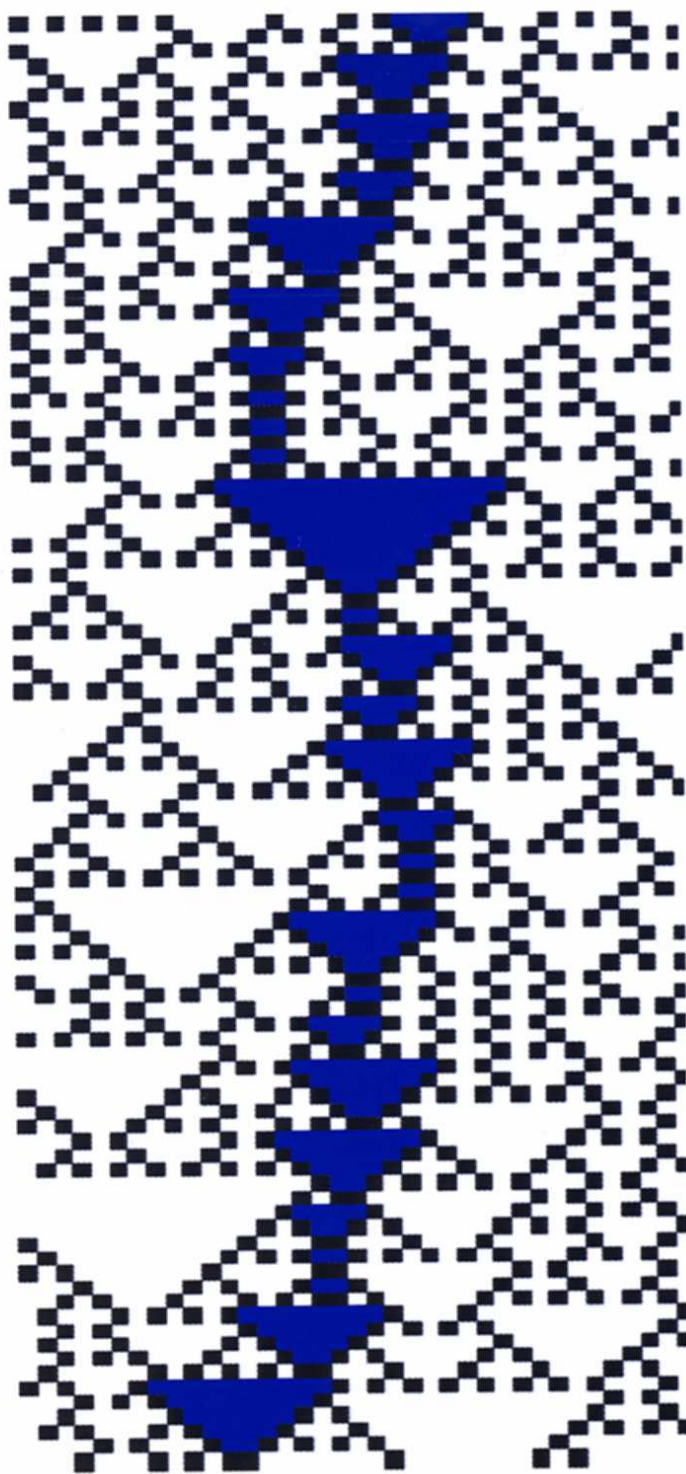


Figure 4. A Rule 18 “particle” drifts left and right.

generate patterns that are qualitatively similar regardless of the initial condition chosen. For many rules, a single 1 in a sea of 0’s is enough to spawn a complex pattern.

But computer experiments alone don’t prove anything. For one thing, computations are always done on systems of limited size, but the conjectures assert that the observed behavior holds for *all* systems, no matter how large. In particular, claims of chaos based on computer simulations, while often persuasive, fall short of mathematical rigor, since the behavior of any cellular automaton is ultimately periodic: Because it’s finite, an automaton must eventually repeat one of its states. As soon as that happens, the deterministic rule demands that the system simply cycle endlessly through an unvarying sequence of states—behavior that is the very antithesis of chaos. Computer experiments are also not particularly helpful in developing a theory that explains why rules generate the patterns they do, or why two rules are similar or different in the patterns they generate.

“Trivial” systems, of course, don’t require much proof. For example, rule 240 (= 11110000), which tells each cell to adopt the value of the neighbor on its left, obviously produces diagonal stripes of widths that depend only on the initial pattern of 1’s and 0’s. More interesting rules, however, pose real challenges to mathematicians.

Through detailed analysis, Jen has provided mathematically rigorous proofs for some of the observations about elementary automata. She and colleagues, including Peter Grassberger at the University of Wuppertal in Germany and James Crutchfield at the University of California at Berkeley, have found subtle relationships

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between the behavior of certain “simple” systems and patterns that arise in more complicated automata.

One such relationship connects the patterns produced by Rule 90 with those of Rule 18 and several other rules. While hardly trivial, Rule 90 is relatively simple in that it is “linear”: Each cell’s value changes at each step to the sum (mod 2) of the values on either side of it. Because of linearity, the behavior of Rule 90 can be studied using tools from linear algebra. Jen and others have studied linear automata extensively.

Although Rule 18 is nonlinear, Jen has shown that it is equivalent to Rule 90 in a very precise sense. In particular, given any initial pattern consisting of isolated 1’s separated by odd numbers of 0’s, such as 10100010100000, both rules produce exactly the same subsequent behavior. Jen found a way to track the effect of inserting additional 0’s into a Rule-18 pattern. This makes it possible to transform any initial pattern into one with isolated 1’s and odd-length strings of 0’s, use Rule 90 to “evolve” the modified pattern, and still recover the correct final pattern for Rule 18. The procedure, Jen notes, is reminiscent of a technique known as the inverse scattering method, which is used for solving the partial differential equations that give rise to solitons (see “New Wave Mathematics” in *What’s Happening in the Mathematical Sciences*, Volume 2).

One way to display the equivalence of Rules 18 and 90 without actually inserting the extra 0’s is to color-code the stretches where the 0’s would go. The result can be interpreted as “particles” drifting lazily left and right as they move down the page (see Figure 4). Occasionally two particles collide and annihilate each other; on other occasions three particles coalesce into one. What’s crucial is that the number of particles never increases. In a finite setting, the number of particles obviously stabilizes as soon as the automaton settles into a cycle. In an *infinite* setting, with a finite stretch of 1’s and 0’s embedded in a sea of 0’s, Jen has proved that the number of particles dwindles down to either one or zero, depending on whether the original pattern had an odd or even number of particles. In effect, the infinite stretches of 0’s act like force fields, nudging the particles toward the center of the expanding pattern (see Figure 5, next page).

A number of rules share Rule 18’s mechanism of generating diffusively annihilating particles, and can also be mapped onto Rule 90. Other rules, however, exhibit quite different particle-generating

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mechanisms. Rule 54 (= 00110110), for example, operates on random initial patterns to rapidly produce a “background” pattern consisting of the 4-cell “bricks” 1110 and 0010. Any pattern built exclusively with these two bricks will persist, drifting to the right and toggling the two bricks (see Figure 6). Breaks in the pattern, however, such as an extra 1 or 0 stuck between two bricks, tend to form “domain walls” that persist indefinitely. The domains “communicate” by sending particles back and forth. (See Figure 1, page 70.)

Most researchers concentrate on the behavior of the walls and particles, treating the background as immaterial. By focusing on details of the background, however, Jen has been able to give rigorous proofs for some of the observed behavior. For example, looking again at finite patterns embedded in a sea of 0’s, she has shown that for almost any background with two dislocations, the dislocations will drift infinitely far apart, as if there’s some repulsive force between them. Jen conjectures that this is true in general, but that—like much else about the world, whether continuous or discrete—remains to be proved.

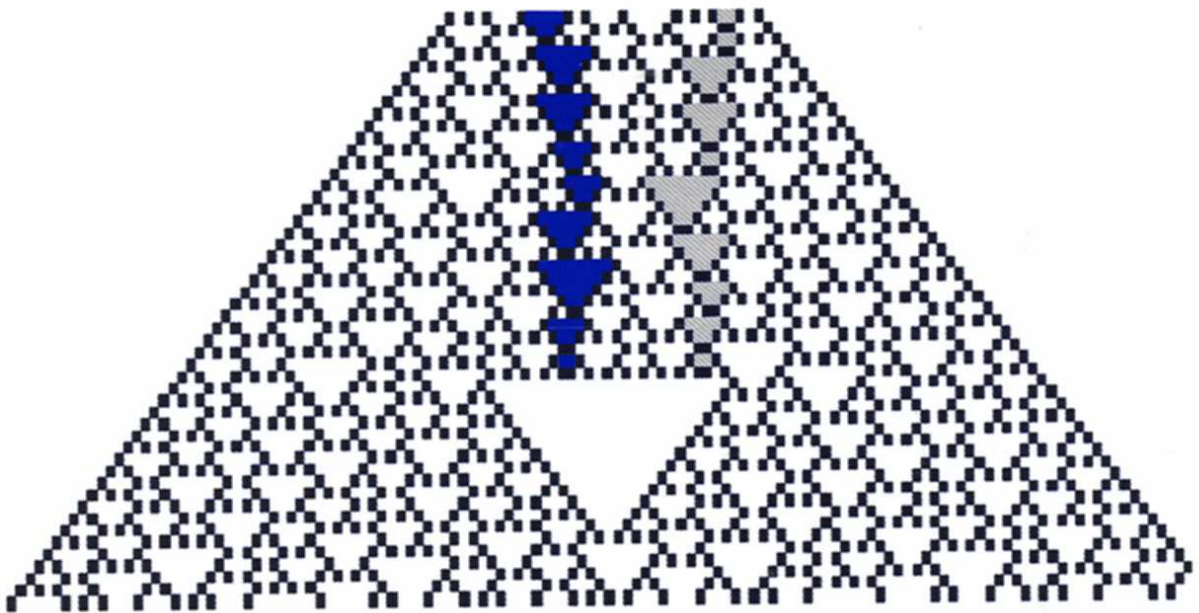
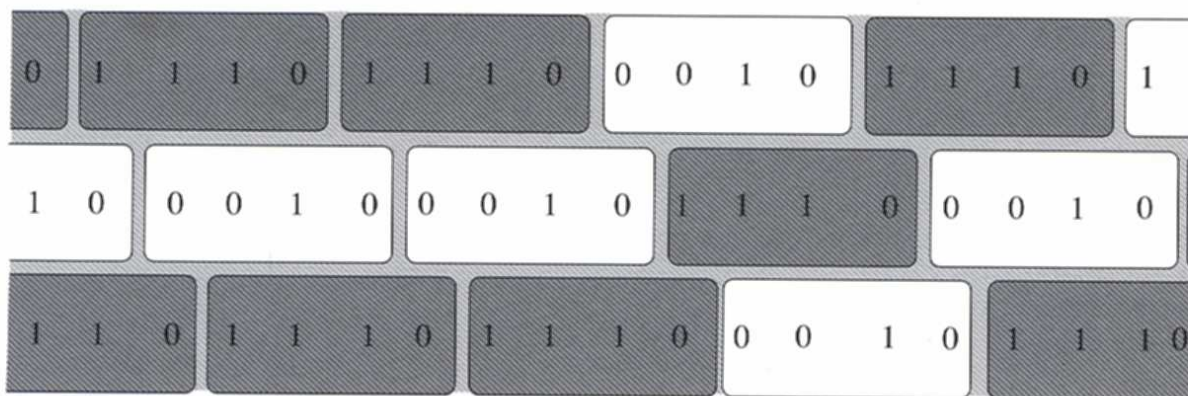


Figure 5. Two Rule-18 particles coalesce and annihilate each other.



**Figure 6.** Rule 54 exhibits simple behavior on any pattern consisting of the two “bricks” 1110 and 0010.

### Taking a Chance with Discrete Mathematics

The deterministic rules of cellular automata are one way to view the world discretely. Another approach is to include an element of chance: Let local conditions determine the probabilities, then do the updates with a computational roll of the dice.

Richard Durrett at Cornell University is betting there’s a payoff in the probabilistic approach. He and colleagues including Simon Levin, a mathematical ecologist at Princeton University, argue that discrete models, coupled with carefully structured random choices, can mirror many complex phenomena observed in biological and environmental systems. These models, which they call interacting particle systems, offer insights into predator–prey competition, bacterial growth, and the spread of epidemics.

A discrete, probabilistic approach seems appropriate for studying biological systems, since life thrives on variability and random selection, and populations consist of individuals. In a forest fire, for example each tree catches fire at a rate proportional to the number of nearby burning trees, while those already on fire burn out at a constant rate. These dynamics can be modeled on a two-dimensional grid (the regular spacing might suggest an orchard rather than a forest) with values *green*, *red*, and *black*. (The colors represent obvious conditions!) At each step, each green tree turns red

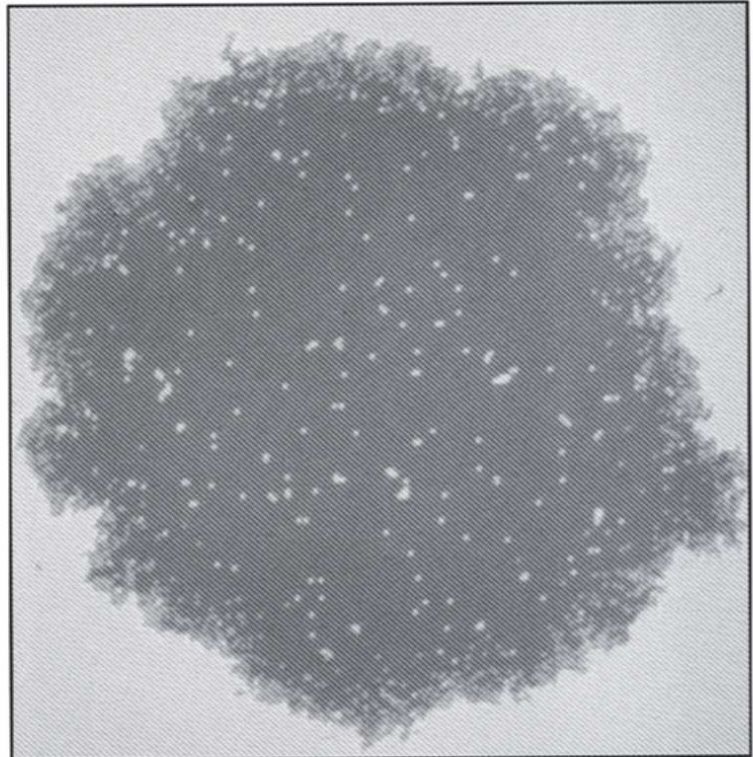
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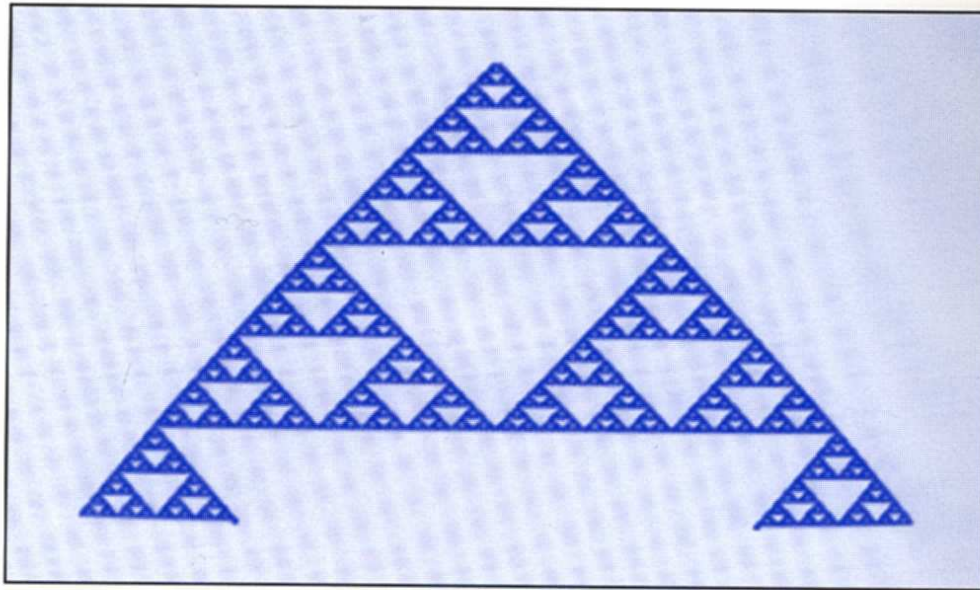
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with probability equal to some fraction of the number of red trees nearby, while red trees turn black with a constant probability. A computational fire that starts in the middle tends to expand in a roughly circular pattern, but with an increasingly convoluted boundary (see Figure 7). It also leaves behind a scattering of unburnt trees; their density might predict how quickly the forest will regenerate.

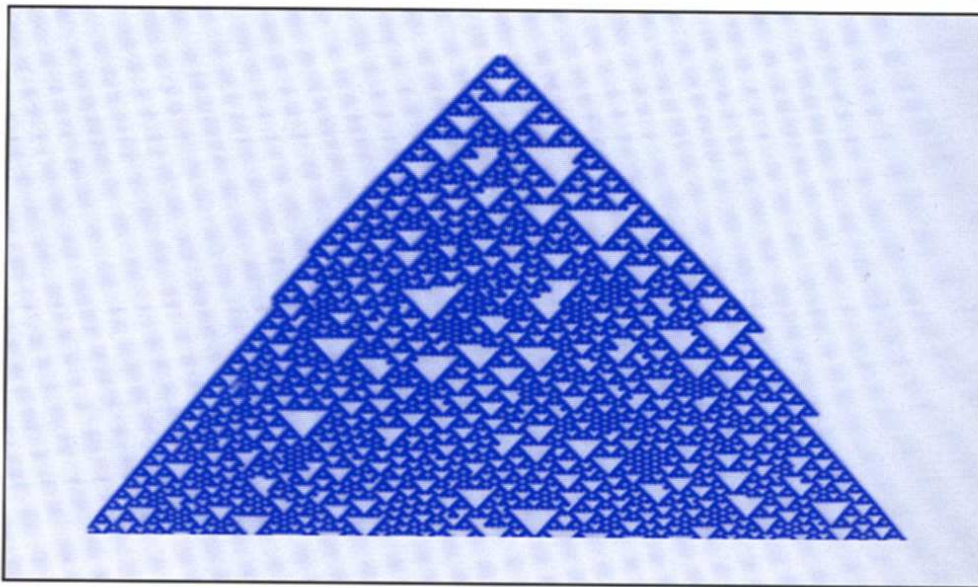
The same basic model, with different parameters, can be used to study the spread of an epidemic or the growth of a bacteria colony. The crude assumptions that define the local interactions are certainly not realistic, notes Durrett. But, he's quick to add, "to my point of view, that's almost an advantage." The models' simplicity makes it possible to identify sources of particular patterns the models have in common. "The more realism you inject in a model, the less confidence you have in what's causing the patterns you're observing," Durrett says. "These things are not meant to *fit* reality, but to try to understand the mechanisms at work in it."



**Figure 7.** A computational forest fire. (Figure courtesy of Richard Durrett, Cornell University.)



**Figure 8a.** *Pascal's triangle mod 2. The familiar fractal-like figure can be interpreted as a one-dimensional, "Rule 60" cellular automaton, with each row shifted half a square to the left.*



**Figure 8b.** *Pascal's triangle mod 2 with random mistakes. Strange things happen if dark squares in each new row are randomly erased (in this case, with probability .02). Oddly enough, random erasures lead to a proliferation of dark squares. (Figures 8a and 8b courtesy of Richard Durrett, Cornell University.)*

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