Solutions to Problem Set 3

May 22, 2001

1. Suppose $\epsilon = \frac{1}{2^n}$, and consider a covering of F with diameter ϵ. This means that if U is an open set in the covering, then the first n letters of each element of U are the same, but the first $n+1$ letters are not all the same. Clearly F can be covered with the 3^n open sets $[\alpha]$, where α runs over all words of length n. But any covering of F with diameter ϵ must have at least 3^n open sets; for each word $[\alpha]$ of length n there must be an open set in the covering all of whose elements start with α. Hence $\mathcal{H}^s_\epsilon = 3^n(\frac{1}{2^n})^s$. If $s = \frac{\log 3}{\log 2}$ then $\mathcal{H}^s_\epsilon = 1$ for $\epsilon = \frac{1}{2^n}$. For any other choice of s, $\mathcal{H}^s_\epsilon$ will approach 0 or ∞ as n approaches ∞ and $\epsilon = \frac{1}{2^n}$ approaches 0. Hence the Hausdorff dimension of F is $\frac{\log 3}{\log 2}$.

2. The set A of points (x, y) for which x and y have binary expansions with no 1's in the same spot is clearly a nonempty subset of the unit equilateral triangle. Note that the transformations T_L, T_U and T_R send (x, y) to $(.0x, .0y)$, $(.0x, .1y)$ and $(.1x, .0y)$. (Hopefully the notation here is clear). Clearly the transformation $K \to T_L(K) \cup T_U(K) \cup T_R(K)$ fixes A.

3. If s and t are a distance $\frac{1}{2^k}$ apart, then they agree in the first k letters. Hence $h(s)$ and $h(t)$ lie in the same equilateral triangle of side length $\frac{1}{2^k}$ determined by those k letters. Thus the Eucildean distance between $h(s)$ and $h(t)$ is at most $\frac{1}{2^k}$.

If $s = LRRRRR\ldots LLL\ldots$ and $t = RLLL\ldots$, where there are n R's between the first L and the tail end in s, then s and t are a distance $\frac{1}{2}$ away, but $h(s)$ and $h(t)$ are a distance $\frac{1}{2^{n+1}}$ away. Hence there exists no constant C between 0 and 1 such that $||h(s) - h(t)|| \geq Cd_{\frac{1}{2}}(s, t)$ for all points s and t. Thus h does not have bounded decrease.

4. Since $||h(s) - h(t)|| \leq d_{\frac{1}{2}}(s, t)$, one sees from the definitions that $\mathcal{H}^s_\epsilon(h(F)) \leq \mathcal{H}^s_\epsilon(F)$. This is simply because every ϵ-covering of F is also an ϵ-covering for the Sierpinski gasket $h(F)$. Since the diameters of the open sets are all less than or equal to 1, this means that the Hausdorff dimension of $h(F)$ is less than or equal to the Hausdorff dimension of F.

5. One sees that $f'(s) = s\Sigma_i \log(r_i) r_i^s$. Since the r_i are less than 1, the $\log(r_i)$ are negative, so $f'(s)$ is negative. By the intermediate value theorem, there exists s such that $f'(s) = 1$. Since f is strictly decreasing, this s is unique.

6. The similarity dimension of the Cantor set is $\frac{\log 2}{\log 3}$. The similarity dimension of the Sierpinski gasket is $\frac{\log 3}{\log 2}$.