Problem set 5, Math 118

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Contents

1 The question.

The simplest non-trivial shift space consists of the space $\{0,1\}^{\mathbf{Z}}$ of all doubly infinite sequences in a binary alphabet. It has entropy 1. In fact, its graph consists of a single vertex with two loops - see the graph L_0 in the figure labeled "Loops allowed" below. Its adjacency matrix is the one-by-one matrix

$$(2)$$

which clearly has 2 as its sole eigenvalue. The purpose of this problem set is to classify all possible shifts with entropy 1, i.e. all finite matrices with non-negative integer entries whose maximal eigenvalue is 2. We will do this subject to two additional conditions. The first condition is that the matrix be irreducible. In terms of the graph this means that we can get from any vertex to any other vertex by some succession of edges. This is an obviously reasonable restriction. The second condition, which is harder to justify, and we won't try, is that the graph have the following property: if there is an edge joining vertex i to vertex j then there is an edge joining vertex j to vertex i. So in terms of the adjacency matrix A this says that if $A_{ij} \neq 0$ then $A_{ji} \neq 0$. It does not say that these two numbers are equal, only that if one is not zero, the other is also not zero. A matrix with this property is sometimes called *symmetrizable.* We shall use this (unfortunate) terminology.

2 Reading the Diagrams

Each of the diagrams below corresponds to a matrix with non-negative integer entries whose rows and columns are parametrized by the vertices of the graph. The entry $A_{ij} \neq 0$ if and only if there is an edge joining the vertices i and j. If there is no arrow pointing towards the vertex i then $A_{ij} = 1$. If there is an arrow pointing towards the vertex i, then $A_{ij} =$ the number of lines joining i to j.

For example, the graph labeled $A_1^{(1)}$ in Table **Aff 1** corresponds to the matrix

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Notice that the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of this matrix with eigenvalue 2. This is the meaning of the numbers attached to the vertices.

In fact, you can check that these integer vectors are indeed eigenvectors with eigenvalue 2, by observing that the value above any node is one-half the sum of the values at the adjacent nodes multiplied by the appropriate valency given by the number of lines pointing toward the node. For example, Diagram $A_2^{(2)}$ in Table **Aff 2** corresponds to the matrix

$$\begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$$

which has $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ as eigenvector with eigenvalue 2. We can read this off the diagram directly by seeing that 2, the value over the first node is one half of 4×1 while 1, the value over the second node is one half of 2. Similarly, in the graph $E_8^{(1)}$, where all the non-zero matrix entries equal 1, we have $1 = \frac{1}{2} \times 2$, $2 = \frac{1}{2} \times (1+3)$, $3 = \frac{1}{2} \times (2+4)$, etc. At the branch we have $6 = \frac{1}{2} \times (5+4+3)$ and so on.

Recall that a matrix A is *symmetrizable* if $A_{ij} \neq 0 \Rightarrow A_{ji} \neq 0$. A matrix with non-negative entries is called *irreducible* if for every i and j there is an n such that $(A^n)_{ij} \neq 0$. Each diagram above gives a symmetrizable irreducible matrix with non-negative integer entries and with 2 as its maximum eigenvalue.

As a third example, $A_3^{(1)}$ corresponds to the four by four matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

The Perron Frobenius theorem says that for an irreducible matrix with non-negative entries, the maximum eigenvalue has an eigenvectgor with all entries positive, and this is the only eigenvalue with this property. Since all the entries above the nodes in the diagrams are non-zero, this shows that 2 is indeed the maximum eigenvalue, and hence each of these graphs corresponds to a shift dynamical system with entropy 1.

The Perron Frobenius theorem also says that if we decrease the matrix by decreasing any of the entries, or by striking out a row and a column with the same index, (say the i-th row and column) and so get a matrix of smaller size, this strictly decreases the maximum eigenvlue. In terms of a graph, this means that if we have a graph Γ_1 which is a strict subgraph of another graph Γ_2 in the sense that it has fewer edges or fewer nodes, then the maximal eigenvalue of the adjacency matrix of Γ_1 is strictly less than that of Γ_2.

For example, this implies that all $A_{ij} \leq 4$ if the maximal eigenvalue of the adjacency matrix is 2 and that $A_2^{(2)}$ is the only such graph which has an $A_{ij} = 4$. Indeed, if a graph had an $A_{ij} \geq 4$ it would have to contain $A_2^{(2)}$ as a subgraph, simply by removing all but the i-th and j-th node, and then decreasing the number of edges joining i to j. If any of these operations really took place, i.e. if $A_2^{(2)}$ was a strict subgraph, then the maximal eigenvalue of the adjacency matrix of the original graph would have had to be stricly larger than 2.

Similarly, $A_1^{(1)}$ is the only graph for which there is a pair of vertices i and j for which both $A_{ij} > 1$ and $A_{ji} > 1$ since any graph with this property would have to contain $A_1^{(1)}$ as a subgraph. So with the exception of $A_1^{(1)}$, we can assume that if $A_{ij} \neq 0$ then either $A_{ij} = 1$ or $A_{ji} = 1$.

Any graph with more than three vertices can not have any $A_{ij} \geq 3$ as it would have to contain either $G_2^{(1)}$ or $D_4^{(3)}$ as sub-graphs, and hence these two are the only graphs without loops having any $A_{ij} \geq 3$, since any graph with two vertices and $A_{12} = 3$ and $A_{21} = 1$ and $A_{11} = A_{22} = 0$ is a strict subgraph of $G_2^{(1)}$ and hence its maximum eigenvalue is strictly less than 2. On the other hand, the graphs LC_ℓ and LB_ℓ show that any graph with a loop and some $A_{ij} > 2$ has maximum eigenvalue > 2.

3 Homework Problem.

Show that the any irreducible symmetrizable matrix with non-negative integre entries and whose maximal eigenvalue is 2 must correspond to one of the graphs drawn below. Use the above kind of reasoning. For example, you could begin by showing that $A_\ell^{(1)}$, $\ell \geq 1$ is the only such graph containing a cycle, that any graph wich is not $A_\ell^{(1)}$ must contain a branch or a loop or have some $A_{ij} > 1$ (or some combination of the above).

4 The diagrams.

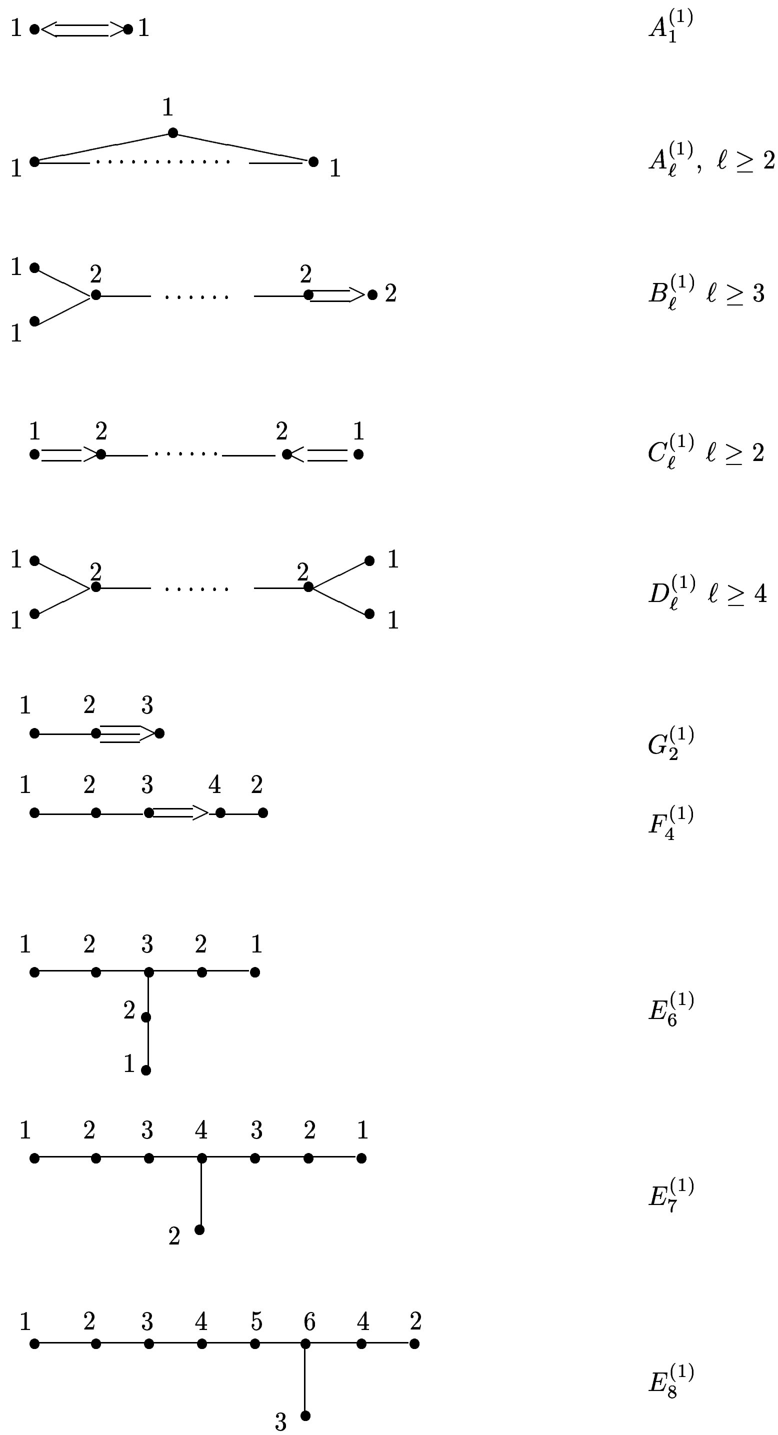

Figure 1: Aff 1.

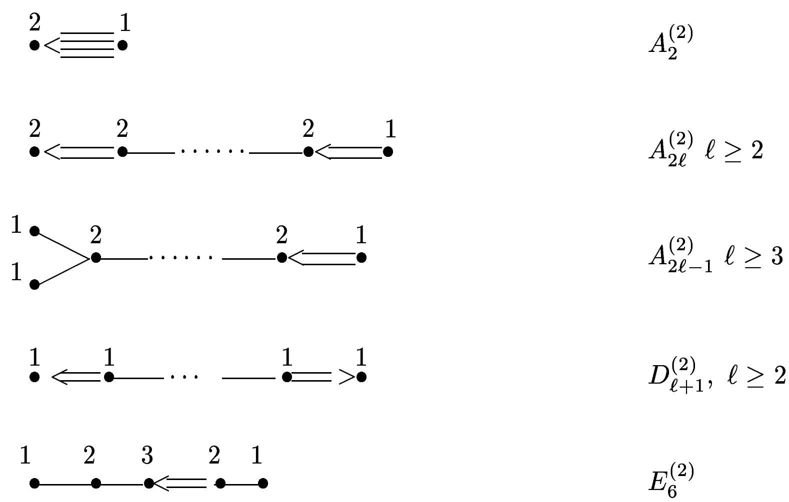

Figure 2: Aff 2

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Figure 3: Aff 3

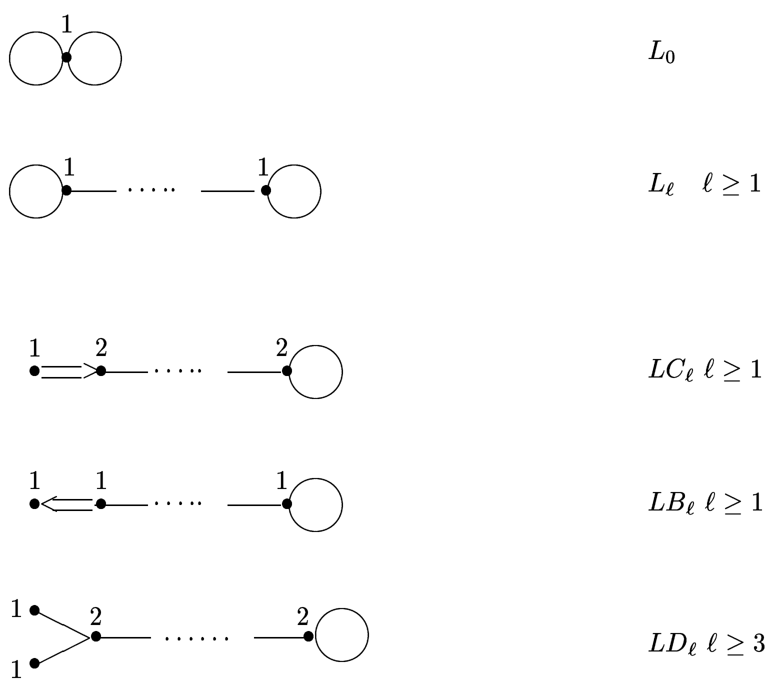

Figure 4: Loops allowed