Solutions to Problem Set 2

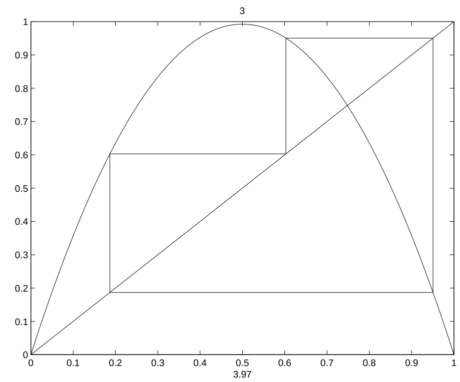
March 15, 2001

1. Set $f(x) = 2x^2 - 5x$. Then $f^{\circ 2}(x) = 8x^4 - 40x^3 + 40x^2 + 25x$. Solving $x = f^{\circ 2}(x)$ yields $x = 0, 3, 1 \pm \sqrt{2}$. The derivative of $f^{\circ 2}$ at $1 + \sqrt{2}$ is -31, so $1 \pm \sqrt{2}$ form a repelling period-2 orbit.

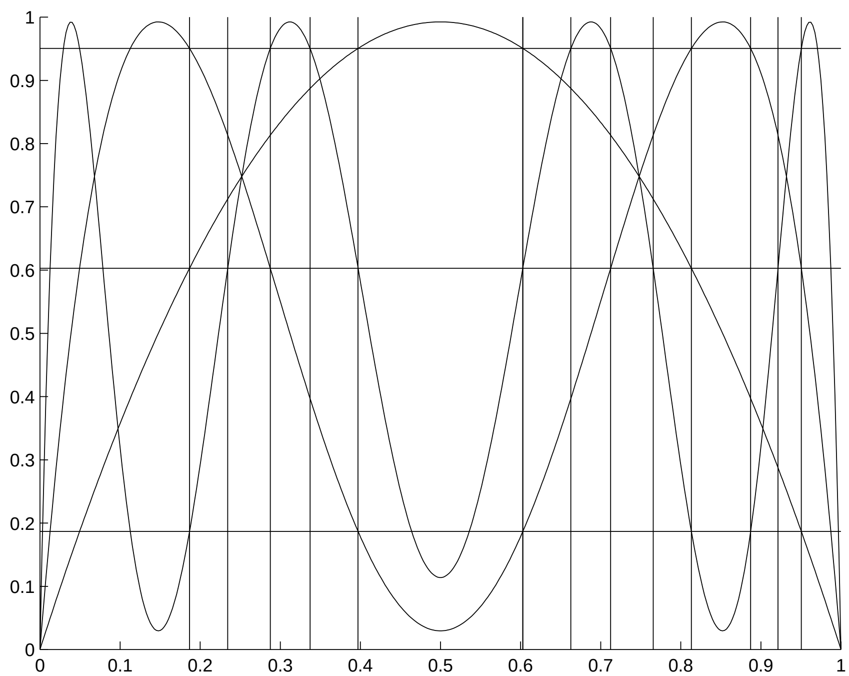
2. The following file was available on the website under the name

`graphiciterationlogistic.m`.

```
%movie of graphical iteration of logistic function
%input number of iterates, parameter and start
close all
clc
r=input('r=')
p=input('initial point =')
iterates =input('number of iterations =')
x=linspace(0,1,400);
y=logistic(x,r);
plot(x, x)
hold on
plot(x,y,'r')
q=logistic(p,r);
for j=1:iterates
plot([p p],[p,q],'k')
plot([p q],[q q],'k')
title(j)
xlabel(r)
p=q;
q=logistic(p,r);
pause(3)
end
zoom
```



3. The vertical lines in the figure below divide the unit interval up into subintervals, some of which are gaps which eventually map out of $[A, C]$. From left to right, they are: gap, $LRRL$, $LRRR$, gap, $LRLR$, gap, $RRLR$, gap, $RRRR$, $RRRL$, gap, $RLRL$, $RLRR$, gap. The vertical and horizontal lines are superimposed upon graphs of $L_{3.97}$, $L_{3.97}^{\circ 2}$ and $L_{3.97}^{\circ 3}$.



Since $f(L) = R$ and $f(R) = L \cup R$, $L's$ must be followed by $R's$, but $R's$ can be followed by $L's$ or $R's$.

4. First, we should know that the set of points following any itinerary $S_1 S_2 \ldots S_n$ is in fact an interval; this follows from the unimodality. (We should also remark that unimodality means that the function has a unique critical point, not just a unique maximum.) It was not necessary to prove that each $S_1S_2\dots S_n$ was an interval; bonus points were awarded to those that did so.

Notice that the interval RR lies between the intervals RL and LR, so that all points in RL are at least a distance d away from all points in LR. If J is an itinerary ending in R, clearly J contains the subintervals JRL and JLR. After $k+1$ iterations, points in JRL end up in RL and points in JLR end up in LR. Hence after $k+1$ iterations, points in JRL and JLR end up a distance at least d away from each other.

5. Inside J lie the intervals $S_1S_2\dots S_kRRL$, $S_1S_2\dots S_kRRR$ and $S_1S_2\dots S_kRLR$. These intervals all have positive length. If two of them had length greater than or equal to $\frac{\text{length}(J)}{2}$, then the total length of all three of them would be greater than length(J), a contradiction. Hence at most one of them can have length greater than $\frac{\text{length}(J)}{2}$, and thus at least two of them have length less than this.

6. By problem 5, we may pick $S_{k+2}S_{k+3}$ such that $S_1S_2\dots S_kRS_{k+2}S_{k+3}$ has length less than $\frac{\text{length}(R)}{2}$. Inside $S_1S_2\dots S_kRS_{k+2}S_{k+3}$ are the intervals $S_1S_2\dots S_kRS_{k+2}S_{k+3}RLR = J_1$ and $S_1S_2\dots S_kRS_{k+2}S_{k+3}RRL = J_2$. Since these are subintervals of $S_1S_2\dots S_kRS_{k+2}S_{k+3}$, all points in J_1 are a distance less than length(J) away from all points in J_2.

7. By problem 5, there exist S_2S_3 and T_2T_3 such that the intervals RS_2S_3 and RT_2T_3 have length less than half the length of R. Hence they have length less than $\frac{h}{2}$. Repeating this process, there exist S_2S_3, S_4S_5, $\dots$, $S_{2k}S_{2k+1}$ and $T_2T_3, \dots, T_{2k}T_{2k+1}$ such that $RS_2S_3RS_4S_5\dots RS_{2k}S_{2k+1} = X$ and all the other intervals with any of the $S_{2i}S_{2i+1}$ replaced with $T_{2i}T_{2i+1}$ have length less than $\frac{h}{2^k}$.

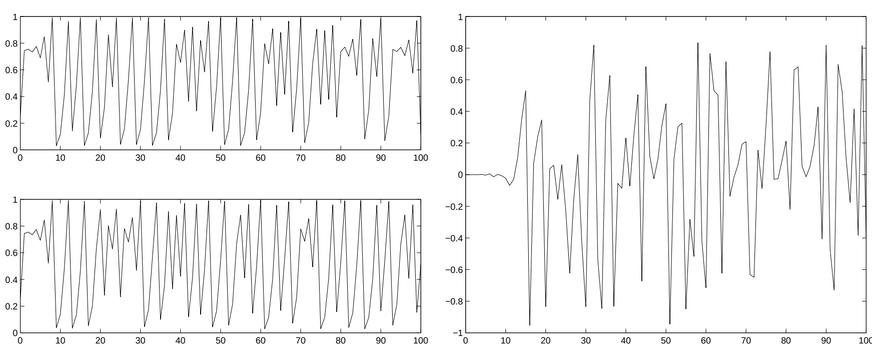
The interval $XRLR$ has a neighbor $XRRL$ which is a distance less than $\frac{h}{2^k}$ away. After $2k+2$ iterations $XRLR$ maps into LR and $XRRL$ maps into RL, and LR and RL are a distance at least d away. A similar result holds with any of the $S_{2i}S_{2i+1}$ replaced by $T_{2i}T_{2i+1}$, and there are 2^k such choices.

8. Here's the program I wrote:

```
%this program calls for x,y, and a number of iterates N
%and plots N iterations of the logistic function for x and y,
%as well as their differences
x=input('Enter an initial value x: ');
y=input('Enter an initial value y: ');
mu=input('Enter the parameter mu for the logistic function: ');
N=input('Enter the number of iterations N: ');

X=zeros(1,N+1);
Y=zeros(1,N+1);

X(1)=x;
```

```
Y(1)=y;

for i=1:N
X(i+1) = logistic(X(i),mu);
Y(i+1) = logistic(Y(i),mu);
end

subplot(2,1,1), plot(0:N,X)
     subplot(2,1,2), plot(0:N,Y)
figure
plot (0:N,(X-Y))
```

The figures on the left plot $L^{\circ n}_{3.97}(.2501)$ and $L^{\circ n}_{3.97}(.2501)$ versus n. The figure on the right plots their difference versus n.



9. One easily checks that $F^{\circ 3}$ maps $[1,2]$ into $[2,3]$, $[2,3]$ into $[3,5]$, and $[4,5]$ into $[1,4]$. Since 1,2,3,4 and 5 are points of period 5, they are certainly not points of period 3. Hence the only possible points of period 3 lie in $(3,4)$. One then checks that $F^{\circ 3}$ is monotonically decreasing on $(3,4)$ and therefore has at most one fixed point. But F has the fixed point $\frac{10}{3}$, so the fixed point of $F^{\circ 3}$ must be $\frac{10}{3}$. Hence F has no points of period 3.

10. Suppose x is a periodic point of Wf.

Case 1. $x \in [0, \frac{1}{3}]$. Wf maps $[0, \frac{1}{3}]$ into $[\frac{2}{3}, 1]$, so x must have even period. Now $Wf^{\circ 2}$ maps x into $\frac{f(3x)}{3}$. If x is a periodic point of $Wf^{\circ 2}$ with period k, i.e. x is a periodic point of Wf with period $2k$, then $3x$ is a periodic point of f with period k and vice-versa. (This can be shown by noting that f and $Wf^{\circ 2}$ are conjugate to each other via the map $x \to y = 3x$.) Hence the periodic points x of Wf in $[0, \frac{1}{3}]$ are precisely the points of the form $\frac{y}{3}$ where y is a periodic point of f. If y has period k, then x has period $2k$.

Case 2. $x \in [\frac{2}{3}, 1]$. As above, x must have even period. $Wf^{\circ 2}$ maps x to $\frac{f(3x-2)+2}{3}$. Again, $Wf^{\circ 2}$ and f are conjugate to each other, this time via the map $x \to y = 3x - 2$. Hence the periodic points x of Wf are of the form $\frac{y+2}{3}$, where y is a periodic point of f. Again, if y has period k then x has period $2k$.

Case 3. $x \in [\frac{1}{3}, \frac{2}{3}]$. Clearly there is a unique fixed point here. Because the slope of Wf is less than -1 here, any point not equal to the fixed point will move away and eventually be mapped into either $[0, \frac{1}{3}]$ or $[\frac{2}{3}, 1]$. Hence the only periodic point here is the unique fixed point.

Given this construction, we note that the function x has only points of period 1. Hence the function $W^j x$ has points of periods 1, 2, 4, ..., 2^j but no point of period 2^{j+1}.