Problem 1.10.7

Let $f : [0,1] \to [0,1]$. Define the double of f, $F : [0,1] \to [0,1]$ in the following way

$$F(x) = \begin{cases} f(3x)/3 + 2/3, & 0 \le x \le 1/3 \\ (f(1)+2)(2/3 - x), & 1/3 \le x \le 2/3 \\ x - 2/3, & 2/3 \le x \le 1 \end{cases} . \tag{1}$$

It is easy to verify this matches with graphical definition given in the text.

Claim. A point x is f-periodic of period n if and only if $x/3$ is F-periodic of period $2n$.

Proof. First let us show that

$$F^{2m-1}(x) = \frac{1}{3} f^m(3x) + \frac{2}{3}, \tag{2}$$

for any integer m. We show this by induction. Eq. (2) is apparent for $n = 1$, for $0 \le x \le 1/3$, $2/3 \le F(x) \le 1$; $F^2(x) = 1/3 f(3x) + 2/3 - 2/3 = 1/3 f(3x)$, $2/3 \le F^2(x) \le 1$; $F^3(x) = 1/3 f(3(1/3 f(3x))) + 2/3 = 1/3 f^2(3x) + 2/3$. Suppose it is true for $m-1$, $F^{2(m-1)-1}(x) = 1/3 f^{m-1}(3x) + 2/3$ and $F^{2(m-1)}(x) = 1/3 f^{m-1}(3x)$. By induction $F^{2m-1}(x) = F^2(F^{2(m-1)-1}(x)) = F^2(1/3 f^{m-1}(3x) + 2/3)$. Now since this $1/3 f^{m-1}(3x) + 2/3 \ge 2/3$, it follows that $F^{2m-1}(x) = F(1/3 f^{m-1}(3x)) = 1/3 f(3(1/3 f^{m-1}(3x))) + 2/3 = 1/3 f^m(3x) + 2/3$, $2/3 \le F^{2m-1} \le 1$. Thus claim of Eq. (2) is shown.

Now consider a periodic p with prime period n, $f^n(p) = p$. Then

$$F^{2n}(p/3) = F(F^{2n-1}(p/3)) = F(\frac{1}{3} f^n(3(p/3)) + 2/3) = F(p/3 + 2/3) = p/3\,, \tag{3}$$

where the last equality holds because of the third condition of the piece-wise definition.

Similarly, we know that $f^m(p) \ne p$ means that

$$\begin{aligned} F^{2m}(p/3) &= F(F^{2m-1}(p/3)) = F(\frac{1}{3} f^m(3(p/3)) + 2/3) = F(\frac{1}{3} f^m(p) + 2/3) \\ &= f^m(p)/3 \ne p/3\,, \end{aligned} \tag{4}$$

and also

$$F^{2m-1}(p/3) = F^{2m-1}(p/3) = \frac{1}{3} f^m(3(p/3)) + 2/3 \ne p/3\,, \tag{5}$$

because $p/3 \le 1/3$ whereas $1/3 f^m(3(p/3)) + 2/3 \ge 2/3$. This shows that $F^m(p/3) \ne p/3$ for any $m < n$.

The reverse implication is similarly derived because our inductive definition of F holds in both directions. (Assume $p/3$ periodic of period $2n$ for F, i.e. $F^{2n}(p/3) = p/3$. Then

$$F^{2n+1}(p/3) = F(p/3) = \frac{1}{3} f(p) + 2/3 = \frac{1}{3} f^{2n+1}(p) + 2/3\,, \tag{6}$$

$f(p) = f^{n+1}(p)$, thus $p = f^n(p)$.)

Problem 1.10.8

Take $f(x)$: $[0,1] \to [0,1]$, $f(x) = 1 - x$. For any $x \neq 1/2$, x is periodic of period 2, $x = 1/2$ is a fixed point. Therefore there are no periodic points of period greater than 2 (by 7).

Then the double $F(x)$ of $f(x)$ has periodic points of period $4 = 2^2$ but not $8 = 2^3$ (by 7).

Taking the double $j-1$ times gives us a map that has periodic points of period 2^j but not of period 2^l for $l > j$.

$G^{j-2}(F(f(x)))$: F is the double of $f(x)$ and G is the double of $F(f(x))$. We are able to iterate the same doubling function G $j-2$ times because after the first application of F our function will always give the value $1/3$ at $x = 1$. Therefore the double will look the same on $[1/3, 2/3]$. Since the double already looks the same on $[0, 1/3]$ and $[2/3, 1]$, now after the first application of F the doubling function G will be identical for the next $j-2$ iterations.

Problem 1.10.9

The slope of $F(x)$, $1/3 \leq x \leq 2/3$, is less than -1, hence the fixed point in $[1/3, 2/3]$ is repelling on $[1/3, 2/3]$ (because $F(x)$ is linear on $[1/3, 2/3]$). Therefore there are no periodic points of prime period greater that 1 on $[1/3, 2/3]$, for any p periodic of prime period > 1, $p \in [0, 1/3] \cup [2/3, 1]$, $p \in [2/3, 1]$, hence $F(p) = x - 2/3 \in [0, 1/3]$.

If $p \in [0, 1/3]$, $F^n(p) = p$ then (by 7) n is odd and $F^n(p) = p = 1/3 f^{(n+1)/2}(3p) + 2/3$, hence $F^{n+1}(p) = F(p) = 1/3 f(3p) + 2/3 = 1/3 f^{(n+1)/2}(3p) + 2/3$. Therefore $f(3p) = f^{(n+1)/2}(3p)$, i.e. $3p = f^{(n-1)/2}(3p)$. Thus $3p$ is periodic for f.

If $F(p) \in [0, 1/3]$, $F^{n+1}(p) = F(p)$, then n is odd and $F^{n+1}(p) = F(p) = 1/3 f^{(n+1)/2}(3F(p)) + 2/3$, hence $F^{n+1}(F(p)) = F(F(p)) = 1/3 f(3F(p)) + 2/3 = 1/3 f^{(n+1)/2}(3F(p)) + 2/3$. Therefore $f(3F(p)) = f^{(n+1)/2}(3F(p))$, i.e. $3F(p) = f^{(n-1)/2}(3F(p))$. Thus $3F(p)$ is periodic for f.

Additional problem 1 (Möbius transform)

Consider the Taylor Expansion at 0 of the function $f(x)$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + o(x^3)\,. \tag{7}$$

(We use Mathematica to perform the algebra for the rest of this section.) Now if we substitute this into our expression, we get (omitting the limit),

$$\begin{aligned} \frac{\phi \circ f(x) - x}{x^3} &= \frac{(a - xc)f(x) + b - xd}{x^3(cf(x) + d)} \\ &= \big[(b + af(x)) + (-d - cf(0) + af'(0))x + (-cf'(0) + af''(0)/2)x^2 \\ &+ (-f''(0)/2 + af'''(0)/6)x^3 + o(x^3)\big] \,/\, \big[x^3(cf(x) + d)\big]\,. \end{aligned} \tag{8}$$

Now we observe that this limit can only exist when the coefficient of the terms of degree 0 through 2 are zeros. We can solve this system of three equations and obtain the following equalities:

$$b = -af(0)\,,\ c = a\frac{f''(0)}{2f'(0)}\,,\ d = a\left(f'(0) - \frac{f(0)f''(0)}{2f'(0)}\right)\,. \tag{9}$$

Now we substitute Eq. (9) into the limit, Eq. (8),

$$\lim_{x\to 0} \frac{a\left[f'''(0)/6 - f''(0)^2/4f'(0)\right]x^3 + o(x^3)}{(cf(x)+d)x^3}\,, \tag{10}$$

which can be simplified

$$\lim_{x\to 0} \frac{a\left[f'''(0)/6 - f''(0)^2/4f'(0)\right]x^3}{x^3} \lim_{x\to 0} \frac{1}{cf(x)+d}\,, \tag{11}$$

where we truncated the terms higher than the third power. Substitute c and d, Eq. (9), into the second limit of Eq. (11), we get $1/af'(0)$. Combining two limits, Eq. (11), we have

$$\begin{aligned}\lim_{x\to 0} a\left(\frac{1}{6}f'''(0) - \frac{f''(0)^2}{4f'(0)}\right)\frac{1}{af'(0)} = \frac{f'''(0)}{6f'(0)} - \frac{f''(0)^2}{4f'(0)^2} &= \frac{1}{6}\left(\frac{f'''(0)}{f'(0)} - \frac{3}{2}\left(\frac{f''(0)}{f'(0)}\right)^2\right) \\ &= \frac{Sf(0)}{6}\,. \end{aligned} \tag{12}$$

Note that this setting of b, c, d forces a unique Möbius Transform ϕ, since

$$\begin{aligned}\phi(x) &= \frac{ax+b}{cx+d} = \frac{ax - af(0)}{af''(0)x/2f'(0) + a\left[f'(0) - f(0)f''(0)/2f'(0)\right]} \\ &= \frac{x - f(0)}{f''(0)x/2f'(0) + \left[f'(0) - f(0)f''(0)/2f'(0)\right]}\,, \end{aligned} \tag{13}$$

so our choice of a does not matter.

Additional problem 2 (One-side Attracting Points)

We have a periodic point p that is one-sided attracting for f. Let m be the period of p, and define $g(x) = f^m(x)$. Suppose *w.l.o.g.* that the attracting side is on the right, that is $W(p) = [p, b_0)$ where $b_0 > p$ and $W(p)$ is the stable interval defined the same as for two-sided attracting points. We note that the same definition works since p is the infimum of points that are forward asymptotic to p. The case for $W(p) = (a_0, p]$ is symmetric and we omit the proof.

Claim 1 $g(W(p)) \subset W(p)$

Proof We know that by the definition of stable interval, $\forall x \in g(W(p))$, $g^i(x) \to p$. Since we assume that f is continuous, this means g is continuous and g must take open intervals to open intervals. This immediately gives us that $g^i(x) \subset W(p)$, which means that

$g(W(p)) \subset W(p)$. $\square$

The above claim gives us that $g([p, b_0]) \subset [p, b_0]$, which implies $g(b_0) \in [p, b_0]$. Since b_0 is the boundary of an interval, it must map to a boundary, so either $g(b_0) = p$ or $g(b_0) = b_0$. And we already know from the definition of periodic point that $g(p) = f^m(p) = p$.

Case 1: $g(b_0) = p$

We have $g(p) = g(b_0)$, and so by the Mean Value Theorem there exists $c \in (p, b_0)$ such that $f'(c) = 0$, so there is a critical point of g in $W(p)$.

Case 2: $g(b_0) = b_0$

We know that for p to be one-sided attracting on the right, this means that $|g'(p)| = 1$ and $\exists \varepsilon > 0$ such that $\forall x \in (p, p + \varepsilon)$, $|g'(x)| < 1 \implies f'(x) < 1$. Now because $[g(b_0) - g(p)]/[b_0 - p] = 1$, then by the Mean Value Theorem, there exists $d \in (p, b_0)$ such that $g'(d) = 1$. Now fix some $e < d$ such that $e \in (p, p + \varepsilon)$ and thus $g'(e) < 1$.

Since $|g'(p)| = 1$, this means either $g'(p) = 1$ or $g'(p) = -1$. If $g'(p) = -1$, then since $g(d) = 1$, by the Intermediate Value Theorem there must be some point $c \in (p, d) \subset W(p)$ such that $g'(c) = 0$. If $g'(p) = 1$, then we notice that there is an $x, p < e < d$ such that $g'(e) < 1$. Since $Sf(x) < 0\ \forall x$, this implies $Sg(x) < 0\ \forall x$, and g' can have no positive minima, that means g' must go below 0 between p and d, and again by the Intermediate Value Theorem it must cross 0 somewhere, hence $\exists c \in (p, d) \subset W(p)$ such that $g'(c) = 0$.

This means that g' has a critical point c in $W(p)$. Now by the chain rule we have

$$g'(c) = (f^m)'(c) = f'(f^{m-1}(c))f'(f^{m-2}(c))...f'(f(c))f'(c)\,. \tag{14}$$

Therefore $\exists i$, $0 \leq i \leq m - 1$ such that $f(f^i(c)) = 0$, and $f^i(c)$ is a critical point of f. We know that $f^i(c) \in W(f^i(p))$ because otherwise $f^i(c) \not\to f^i(p) \implies c \not\to f(p)$ a contradiction. Thus there is a critical point of f in $W(f^i(p))$.