

### ANSWER TO SELECTED PROBLEMS IN HOMEWORK SETS 3

§1.5:6b We want to show that for  $|x| > \sqrt{\lambda+1}$ , we have  $|F^n(x)| \rightarrow \infty$  as  $n \rightarrow \infty$ . Notice that  $F(x) = x^3 - \lambda x$  has three fixed points:  $0, -\sqrt{\lambda+1}, \sqrt{\lambda+1}$ . By the (anti)symmetry of  $F(x) = x^3 - \lambda x$  about the origin, we only need to prove for  $x > \sqrt{\lambda+1}$ . For  $x > \sqrt{\lambda+1}$ , we have

$$F(x) = x^3 - \lambda x = x(x^2 - \lambda) > x(1 + \lambda - \lambda) = x.$$

Since for  $x \geq \sqrt{\lambda+1}$ , we have  $F'(x) = 3x^2 - \lambda \geq 3(\lambda+1) - \lambda > 0$ ,  $F$  is increasing on the interval  $[\sqrt{\lambda+1}, \infty)$ . Therefore from  $F(x) > x > \sqrt{\lambda+1}$ , we get  $F^2(x) > F(x) > x > \sqrt{\lambda+1}$ , and inductively,

$$F^{n+1}(x) > F^n(x) > \cdots > F(x) > x \quad \text{for } x > \sqrt{\lambda+1}.$$

So  $F^n(x) \rightarrow \infty$ , as  $n \rightarrow \infty$ , since there is no fixed points of  $F$  in the interval  $(\sqrt{\lambda+1}, \infty)$ . QED.

§1.5:6c Let the set of points which do not tend to  $\infty$  be  $\Lambda$ . From §1.5:6b, we know  $\Lambda \subset [-\sqrt{\lambda+1}, \sqrt{\lambda+1}]$ . Let  $I = [-\sqrt{\lambda+1}, \sqrt{\lambda+1}]$ . Set  $F'(x) = 3x^2 - \lambda = 0$  gives us critical points  $c = \pm\sqrt{\frac{\lambda}{3}}$ . We see that if  $\lambda > 27$ , we have

$$|F(\pm\sqrt{\frac{\lambda}{3}})| = \frac{2}{3}\lambda\sqrt{\frac{\lambda}{3}} > 2\lambda > \lambda + 1 > \sqrt{\lambda+1}.$$

Therefore if  $\lambda > 27$ , the set  $A_0 = \{x \in I \mid |F(x)| > \sqrt{\lambda+1}\}$  is not empty, and  $I - A_0 = I_1 \cup I_2 \cup I_3$ , where  $I_i, i = 1, 2, 3$  are closed intervals. Lets find the end points of  $I_i$ . This can be done by solving  $F(x) = x^3 - \lambda x = \sqrt{\lambda+1}$  and  $F(x) = -\sqrt{\lambda+1}$ . We only solve the first equation. The roots of the second equation can be obtained from the first by the symmetry of  $F$ . Notice that  $x = \sqrt{\lambda+1}$  is a root of

$$(1) \quad x^3 - \lambda x = \sqrt{\lambda+1}.$$

Divide  $x^3 - \lambda x - \sqrt{\lambda+1}$  by  $x - \sqrt{\lambda+1}$ , we factorize  $x^3 - \lambda x - \sqrt{\lambda+1}$  into

$$x^3 - \lambda x - \sqrt{\lambda+1} = (x - \sqrt{\lambda+1})(x^2 + x\sqrt{\lambda+1} + 1)$$

So the roots of (1) are

$$x_{1,2} = \frac{-\sqrt{\lambda+1} \pm \sqrt{\lambda-3}}{2}, \quad x_3 = \sqrt{\lambda+1}$$

and

$$I_1 = [-\sqrt{\lambda+1}, \frac{-\sqrt{\lambda+1} - \sqrt{\lambda-3}}{2}], \quad I_2 = [\frac{-\sqrt{\lambda+1} + \sqrt{\lambda-3}}{2}, \frac{\sqrt{\lambda+1} - \sqrt{\lambda-3}}{2}],$$

and  $I_3 = [\frac{\sqrt{\lambda+1} + \sqrt{\lambda-3}}{2}, \sqrt{\lambda+1}]$ . Now we want to show that if  $\lambda$  is large enough, we have

$$|F'(x)| > 1 \quad \text{for all } x \in I_1 \cup I_2 \cup I_3.$$

Notice that  $F'(x) = 3x^2 - \lambda$ ,  $F''(x) = 3x$ . So  $F'(x)$  is decreasing on  $(-\infty, 0]$ . Therefore  $F'(x_2) > 1$ ,  $F'(x_1) < -1$ , will give that

$$(2) \quad |F'(x)| > 1 \quad \text{for all } x \in I_1 \cup I_2 \cap (-\infty, 0]$$

Now for  $\lambda > 27$ , we have  $\lambda - 3 > \frac{8}{9}\lambda$ , therefore

$$F'(x_1) = 3\left(\frac{-\sqrt{\lambda+1} + \sqrt{\lambda-3}}{2}\right)^2 - \lambda = \frac{1}{2}\lambda - \frac{3}{2} - \frac{3}{2}\sqrt{\lambda+1}\sqrt{\lambda-3} < \frac{1}{2}\lambda - \frac{3}{2} - \sqrt{2}\lambda < -1,$$

and

$$F'(x_2) = 3\left(\frac{-\sqrt{\lambda+1} - \sqrt{\lambda-3}}{2}\right)^2 - \lambda = \frac{1}{2}\lambda - \frac{3}{2} + \frac{3}{2}\sqrt{\lambda+1}\sqrt{\lambda-3} > \frac{1}{2}\lambda - \frac{3}{2} > 1.$$

This shows that (2) holds if  $\lambda > 27$ . From the symmetry of  $F$ , we conclude that for  $\lambda > 27$ , we have

$$|F'(x)| > 1 \quad \text{for all } x \in I_1 \cup I_2 \cup I_3.$$

From here on, we can proceed as in the text book of Devaney. I will omit all the rest, since most of you got them right. QED.