ANSWER TO SELECTED PROBLEMS IN HOMEWORK SETS 1 AND 2

§1.3:7 Assume that $f : R \to R$ is a homeomorphism. Therefore f is either strictly increasing or strictly decreasing. This implies that $f^2 : R \to R$ is strictly increasing. We only need to show that a strictly increasing function $g : R \to R$ does not have periodic points of prime period greater than 1. This will imply that a homeomorphism f doesn't have periodic points of prime period greater than 2.

If not, assume that p is a periodic point of g of prime period $n > 1$, that is $p \neq g(p)$, but $p = g^n(p)$. There are two cases.

1. If $p < g(p)$. Since g is strictly increasing, we have $g(p) < g^2(p)$, this gives $g^2(p) < g^3(p)$, inductively, we get

$$p < g(p) < g^2(p) < \dots < g^{n-1}(p) < g^n(p) < \dots$$

This contradicts with the assumption that $p = g^n(p)$.

2. If $p > g(p)$. Same argument gives

$$p > g(p) > \dots > g^{n-1}(p) > g^n(p) > \dots$$

Again this contradicts with $p = g^n(p)$. QED.

§1.3:8 Prove by contradiction: assume that for some homeomorphism f, there is a point c, not periodic, but for some $m > 0$, $f^m(c)$ is periodic of period n, that is

$$f^{m+n}(c) = f^m(c).$$

Since f is a homeomorphism, we have $f^{m+n-1}(c) = f^{m-1}(c)$, this again implies $f^{m+n-2}(c) = f^{m-2}(c)$. Eventually we get

$$f^n(c) = c.$$

This contradicts with the assumption that c is not periodic. QED

§1.4:3 We want to show that if p is a hyperbolic periodic point of f, there is an open interval $I \ni p$, such that there are no priodic points of f in I other than p. If not, there is a sequence of periodic points p_i of f, such that $p_i \to p$ as $i \to \infty$. From §1.3:7, we know that p_i and p must be fixed points of f^2. That is

$$f^2(p) = p, \quad \text{and } f^2(p_i) = p_i, \qquad \text{for all } i.$$

Since f is a diffeomorphism, we know $(f^2)'(p)$ exist and

$$(f^2)'(p) = \lim_{x \to p} \frac{f^2(x) - f^2(p)}{x - p} = \lim_{i \to \infty} \frac{f^2(p_i) - f^2(p)}{p_i - p} = \lim_{i \to \infty} 1 = 1.$$

This contradicts with the hyperbolicity of p. QED

§1.6:4 We want to show that the set of periodic points of T_2 is dense in $[0,1]$. i.e., any nonempty open interval $(a,b) \subset [0,1]$ contains at least one periodic point of T_2. From §1.6:3, we know that for any $n \in N$, and any $i = 0, 1 \ldots 2^n - 1$, there is at least one periodic point of T_2 in the interval $[\frac{i}{2^n}, \frac{i+1}{2^n}]$. Since $\lim_{n \to \infty} \frac{1}{2^n} = 0$, we can choose a large enough n, such that

$$\frac{2}{2^n} < b - a.$$

Now let i be the largest integer, satisfying $\frac{i}{2^n} \leq a$. That is

$$\frac{i}{2^n} \leq a \qquad \text{but } \frac{i+1}{2^n} > a.$$

We claim that $\frac{i+2}{2^n} < b$. This is because from $b - a > \frac{2}{2^n}$, we have

$$b > a + \frac{2}{2^n} \geq \frac{i}{2^n} + \frac{2}{2^n}.$$

So

$$[\frac{i+1}{2^n}, \frac{i+2}{2^n}] \subset (a,b)$$

Consequently, there is at least one periodic point of T_2 in (a,b). QED